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DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

M.Sc. MATHEMATICS

II YEAR

TOPOLOGY

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TOPOLOGY

Unit I

Topological spaces : Topological spaces – Basis for a topology – The order topology – The product topology on X x Y – The subspace topology – Closed sets and limit points.

Chapter 2 : Sections 12 to 17

UNIT II

Continuous functions: Continuous functions – the product topology – The metric topology.

Chapter 2 : Sections 18 to 21

UNIT III

Connectedness: Connected spaces- Components and Local Connectedness Chapter 3 : Sections 23& 25.

UNIT IV

Compactness : Compact spaces – Limit Point Compactness – Local Compactness. Chapter 3 : Sections 26 to 29(except 27)

UNIT V

Countability and Separation Axiom: The Countability Axioms – The separation Axioms – Normal spaces – The Urysohn Lemma – The Urysohn Metrization Theorem Chapter 4 : Sections 30 to 34.

Text Books James R. Munkres, Topology (2nd Edition) Pearson Education Pvt.Ltd., New Delhi2002(Third Edition Reprint)

Contents

UNIT – 1

TOPOLOGICAL SPACES

1.1. Topological spaces

The concept of Topological spaces is through out of grew out of the satisfy of the real line and Euclidean space and the study of continuous functions on these spaces. In this section unit we define a topological space and we study a number of spaces of constructing a topology on a set so as to make it into a topological space. We also consider some of the elementary concepts associated with topological spaces. Open and closed sets, limit points and continuous functions are introduced as natural generalisations of the corresponding ideas of real line and Euclidean space.

Definition.

A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties.

(i) $\emptyset$ *and* X are in $\mathcal{T}$

(ii) The union of the elements of any sub collection of $\mathcal{T}$ is in $\mathcal{T}$

(iii) The intersection of elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X function which a topology $\mathcal{T}$ has been specified is called a ***topological space***.

Note. A topological space is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ *on* X but we often omit specific mention of $\mathcal{T}$.

Remark.

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an ***open set*** of X if U belongs to the collection $\mathcal{T}$. Using this terminology, we can say that a topological space is a set X together with a collection of subsets of X, called *open set*, such that $\emptyset$ and X are both open and such that arbitrary unions and finite intersections of open sets are open.

Example 1.

Let $X = \{a, b, c\}$. There are many possible topologies on X. Consider the following topologies in the Figure 1.1.

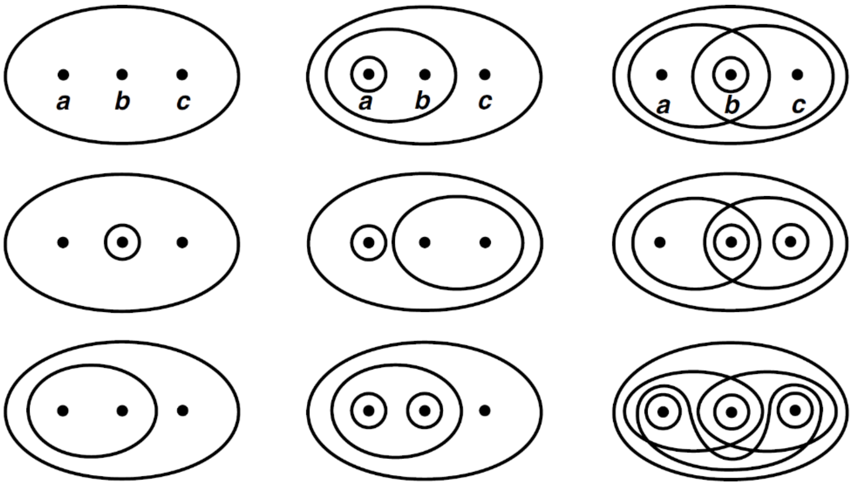


Figure 1.1

The diagram in the upper right-hand corner indicates the topology in which the open sets are $X, \phi, \{a, b\}, \{b\}, and\ \{b, c\}$. The topology in the upper left-hand corner contains only X and ϕ, while the topology in the lower right-hand corner contains every subset of X. We can get other topologies on X by permuting a, b and c.

Note. From the above example, we can see that even a three-element set has many different topologies. But not every collection of subsets of X is a topology on X. For instance, neither of the collections indicated in the Figure 1.2 is a topology.

Figure 1.2

Example 2.

If X is any set, the collection of all subsets of X is a topology on X, then it is called the ***discrete topology***. The collection consisting of X and $\emptyset$ only is also a topology on X, then it is called the ***indiscrete topology*** or the ***trivial topology***.

Example 3.

Let X be set. Let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X \sim U$ either is finite (or) is all of X. Then $\mathcal{T}_f$ is a topology on X, is called the ***finite complement topology***.

For, since $X - X = \emptyset$ is finite (or) $X \sim \emptyset = X$, either is finite or is all of X.

$\therefore$ Both X and $\emptyset$ are in $\mathcal{T}_f$.

Let $\{\cup_\alpha\}$ be an indexed family of non-empty elements of τ_f

To show that $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_f$

Now, $X - \cup\, U_\alpha = \cap\, (X \sim U_\alpha)$

Since each $X \sim U_\alpha$ is finite, $\cap\, (X \sim U_\alpha)$ is finite

$\therefore X \sim \cup\, U_\alpha$ is finite

$\cup\, U_\alpha \in \mathcal{T}_f$

If $U_1, U_2, \dots \dots \dots\ U_n$ are non-empty elements of $\mathcal{T}_f$

To show that $\bigcap_{i=1}^{n} U_1 \in \mathcal{T}_f$

Now, $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$

Since each $(X - U_i)$ is finite

$\bigcup_{i=1}^{n} (X - U_i)$ is finite

$X \sim \bigcap_{i=1}^{n} U_i$

$\therefore \bigcap_{i=1}^{n} U_i \in \mathcal{T}_f$

Thus $\mathcal{T}_f$ is a topology on X and it is called ***finite complement topology.***

Example 4.

*Let X be a set. $\mathcal{T}_C$ be the collection of all subsets of X. Such that $X\sim U$ is either countable (or) is all of X. Then $\mathcal{T}_C$ is a topology on X, is called **countable complement topology** on X.*

For, since $\mathcal{T}_C = \{U \leq X/X\sim U\ is\ countable\ (or)\ X\sim U = X\}$

i.e, the countable complement topology on X is the collection of subset $= \{X\} \cup \{U \leq X/U^C\ is\ countable\}$

clearly, $X \in \mathcal{T}_C$

since, $\emptyset^C = X\sim\emptyset = X$ which is a countable set.

$\therefore \emptyset \in \mathcal{T}_C$

Let $\{U_\alpha\}$ be any arbitrary collection of subsets of X from $\mathcal{T}_C$.

Then U_α^C is countable for each $\alpha \in I$

Now, $(U_{\alpha\in 1} U_\alpha)^C = \bigcap_{\alpha\in I} U_\alpha^C$

The intersection of countable collection of sets is countable $\bigcap_{\alpha\in I} U_\alpha^C$ is countable.

$\therefore (U_{\alpha\in 1} U_\alpha)^C$ is countable.

$\Rightarrow U_{\alpha\in T} U_\alpha \in \mathcal{T}_C$

Let $U_1, U_2, \dots \dots \dots U_n$ be a finite collection of subsets for X from $\mathcal{T}_C$.

Then U_i^C is countable for each $i \in 1,2, \dots \dots n$

$$\left(\bigcap_{i=1}^n U_i\right)^C = \bigcup_{i=1}^n U_i^C$$

Since the finite union of a countable collection of sets is countable, $\bigcup_{i=1}^n U_i^C$ countable.

$\therefore \left(\bigcap_{i=1}^n U_i^C\right)$ is countable

$\Rightarrow \left(\bigcap_{i=1}^n U_i\right) \in \mathcal{T}_C$

Thus $\mathcal{T}_C$ is topology on X, *is called **countable complement topology** on X*

Definition.

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is ***finer*** then $\mathcal{T}$ (or) $\mathcal{T}$ is ***coarser*** then $\mathcal{T}'$. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is ***strictly finer*** then $\mathcal{T}$ (or) $\mathcal{T}$ is ***strictly coarser*** finer then $\mathcal{T}'$ We say $\mathcal{T}$ is ***comparable*** with $\mathcal{T}'$ of either $\mathcal{T}' \supset \mathcal{T}\ (or)\ \mathcal{T} \supset \mathcal{T}'$

1.2. Basis for a Topology

Definition.

If X is a set, a ***basis*** for a topology on X is a collection $\mathfrak{B}$ of subsets of X (called ***basis elements)*** such that

(i) For each $x \in X$, there is at least one basis element B containing x

(ii) If x belongs to the intersection of two basis elements $B_1\ and\ B_2$, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathfrak{B}$ satisfies these two conditions, then we define the ***topology $\mathcal{T}$ generated by $\mathfrak{B}$*** as follows: A subset $U\ of\ X$ is said to open in X (i.e to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B$ and $B \subset U$

Note that each basis element is itself an element of $\mathcal{T}$.

Example 1.

Let $\mathfrak{B}$ be the collection of all circular region (interior of circles) in the plane. Then $\mathfrak{B}$ is a basis for the topology on X.

For, since $\mathfrak{B}$ satisfies both conditions for a basis. The second condition is illustrated in figure 1.2.1. In the topology generated by $\mathfrak{B}$, a subset U of the plane is open if every x in U lies in some circular region contained in U.

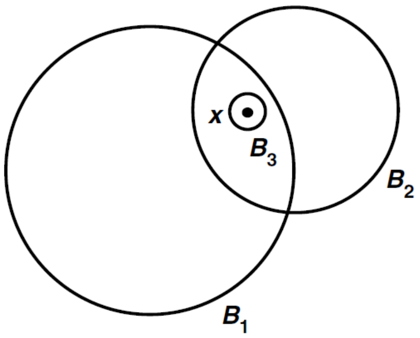
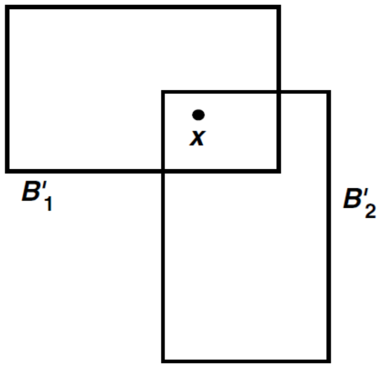


Figure 1.2.1

Example 2.

Let $\mathfrak{B}'$ be the collection of all rectangular regions (ie, the interior of the rectangular) in the plane, where the rectangular have sides parallel to the coordinate axes. Then $\mathfrak{B}'$ is a basis for the topology on X.

For, since $\mathfrak{B}'$ satisfies both conditions for a basis. The second condition is illustrated in figure 1.2.2. In this case, the condition is trivial, because the intersection of any two basis elements us itself a basis element (or empty).



In the topology generated by $\mathfrak{B}'$, a subset U of the plane is open if every x in U lies in some rectangular region contained in U.

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Example 3.

If X is any set, then the collection $\mathcal{T}$ of all one-point subsets of X is a basis for a discrete topology on X and the collection $\mathcal{T}$ generated by the basis $\mathfrak{B}$ is a topology on X.

Solution.

If $U = \emptyset$ then clearly U is open

$\therefore U = \emptyset \in \mathcal{T}$

If for each $x \in X$, there exist a basis element B containing x and $B \subseteq X$

$\therefore X \in \mathcal{T}$

Let us take the indexed family $\{U_\alpha\}_{\alpha \in J}$ of the elements of $\mathcal{T}$

Show that $U = \bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$

Given $x \in U$, there is an index α such that $x \in U_\alpha$.

Since U_α is open, there is a basis element B such that $x \in B \in U_\alpha$.

Then $x \in B$ and $B \subset U$.

∴ By definition, U is open

∴ $U = \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

Next, we show that $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$

Now, let us take two elements U_1 and U_2 of $\mathcal{T}$ and show that $U_1 \cap U_2 \in \mathcal{T}$

Given, $x \in U_1 \cap U_2$, choose a basis of element B_1 containing x such that $x \in B_1 \subset U_1$ and also choose a basis element B_2 containing such that $x \in B_2 \subset U_2$.

Then, by definition, we have to choose a basis element B_3 containing x such that $x \in B_3 \subset B_1 \cap B_2$.

Then $x \in B_3$ and $B_3 \subset U_1 \cap U_2$

$$\therefore U_1 \cap U_2 \in \mathcal{T} \quad \ldots \ldots \ldots \ldots (1)$$

Finally, we show by induction that any finite intersection

$$U_1 \cap U_2 \cap \ldots \ldots \ldots \cap U_n \in \mathcal{T}$$

This fact is trivial when $n = 1$

Suppose it is true for $n - 1$ and prove it for n

Now, $U_1 \cap U_2 \cap \ldots \ldots \ldots \cap U_n = (U_1 \cap U_2 \cap \ldots \ldots \ldots \cap U_{n-1}) \cap U_n$

By induction hypothesis $U_1 \cap U_2 \cap \ldots \ldots \ldots \cap U_{n-1} \in \mathcal{T}$ and by result (1)

$$(U_1 \cap U_2 \cap \ldots \ldots \ldots \cap U_{n-1}) \cap U_n \in \mathcal{T}$$

$$\text{i.e., } U_1 \cap U_2 \cap \ldots \ldots \ldots \cap U_n \in \mathcal{T}$$

∴ The result is true for n

Thus, the collection of open sets generated by a basis $\mathfrak{B}$ is a topology.

Lemma 1.2.1.

Let X be a set. Let $\mathfrak{B}$ be the basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathfrak{B}$.

Proof.

Let X be a set and $\mathfrak{B}$ be the basis for the topology $\mathcal{T}$ on X.

Given a collection of elements of $\mathfrak{B}$, they are also an element of $\mathcal{T}$

Since $\mathcal{T}$ is a topology, then their union is in $\mathcal{T}$

Conversely, given, $U \in \mathcal{T}$

For each $x \in U$, choose an element B_x of $\mathfrak{B}$ $such\ that\ x \in B_x \subset U$.

Then $U = \bigcup_{x \in U} B_x$

Hence U equals a union of elements of $\mathfrak{B}$.

Lemma 1.2.2.

Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of x.

Proof.

We show that $\mathcal{C}$ is a basis

Given $x \in X$. Since X itself an open. Then by hypothesis there is an element C of $\mathcal{C}\ such\ that\ x \in C \subset X$.

Let $x \in C_1 \cap C_2$, where $C_1, C_2 \in \mathcal{C}$

Since C_1 and C_2 are open, $C_1 \cap C_2$ is open.

By hypothesis, there exist an element $C_3\ of\ \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$

$\therefore \mathcal{C}$ is a basis.

Let $\mathcal{T}$ be the collection of open sets of X.

We show that, the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals the topology $\mathcal{T}$.

First note that, if $U \in \mathcal{T}$ and $x \in U$, then by the hypothesis, there is an element $C\ of\ \mathcal{C}$ such that $x \in C \subset U$

$\therefore U \in \mathcal{T}'$

Conversely, if $W \in \mathcal{T}'$, then by lemma 1.2.1 W equals a union of elements of $\mathcal{C}$.

Since, each element of $\mathcal{C}$ belongs to $\mathcal{T}$ and $\mathcal{T}$ is a topology.

$\therefore W \in \mathcal{T}$.

Thus $\mathcal{T} = \mathcal{T}'$

Hence, $\mathcal{C}$ is a basis for the topology of X.

Lemma 1.2.3.

Let $\mathfrak{B}$ and $\mathfrak{B}'$ to the bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. Then the following are equivalent.

i) $\mathcal{T}'$ is finite than $\mathcal{T}$

ii) For each $x \in X$ and each basis element $B \in \mathfrak{B}$ containing x, there is a basis element $B' \in \mathfrak{B}'$. Such that $x \in B' \subset B$.

Proof.

$(ii) \Rightarrow (i)$

Given an element $U \in \mathcal{T}$

We show that $U \in \mathcal{T}'$

Let $x \in U$

Since $\mathfrak{B}$ generates $\mathcal{T}$, there is an element $B \in \mathfrak{B}$ such that $x \in B \subset U$

By (2), there exist basis element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$

Then $x \in B' \subset U$

So, by definition, $U \in \mathcal{T}'$

$\therefore \mathcal{T}'$ is finer than $\mathcal{T}$

$(i) \Rightarrow (ii)$

Given $x \in X$ and $B \in \mathfrak{B}$ with $x \in B$

Then by definition $B \in \mathcal{T}$ and $\mathcal{T}' \supset \mathcal{T}$, by (i)

$\therefore B \in \mathcal{T}'$

Since, $\mathcal{T}'$ is the topology generated by $\mathfrak{B}'$, there is an element $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$

Definition.

If $\mathfrak{B}$ is the collection of open intervals in the real line, $(a, b) = \{x/a < X < b\}$, the topology generated by $\mathfrak{B}$ is called the ***S*** on the real line.

If $\mathfrak{B}'$ is the collection of all half-open intervals of the form $[a, b) = \{x/a \leq x < b\}$, where $a < b$, the topology generated by $\mathfrak{B}'$ is called the ***lower limit topology***. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$.

Let K denote the set of all numbers of the form $\frac{1}{n}$ for $n \in \mathbb{Z}$ and let $\mathfrak{B}''$ be the collection of all open intervals (a, b), along with all lets of the form $(a, b) - K$. The topology generated by $\mathfrak{B}''$ is called the ***K −topology*** on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$.

Lemma 1.2.4.

The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Proof.

Let $\mathcal{T}, \mathcal{T}'$ and $\mathcal{T}''$ be the topologies of $\mathbb{R}, \mathbb{R}_l, and\ \mathbb{R}_K$ respectively.

Given a basis element (a, b) for $\mathcal{T}$ and a point $x \in (a, b)$ the basis element $[x, b)$ for $\mathcal{T}'$ contains x and lies in (a, b). On the other hand, given the basis element $[x, d)$ for $\mathcal{T}'$, there is no open interval (a, b) that contains x and lies on $[a, d)$.

Thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

A similar argument applies to $\mathbb{R}_K$.

Given a basis element (a, b) for $\mathcal{T}$ and a point $x \in (a, b)$, this same interval is a basis element for $\mathcal{T}''$ that contains x. On the other hand, given the basis element $B = (-1,1) - K$ and the point $0\ of\ B$, there is no open interval that contains 0 and lies in B.

sThus $\mathcal{T}''$ is strictly finer than $\mathcal{T}$.

By definition of $\mathbb{R}, \mathbb{R}_l\ and\ \mathbb{R}_K$ topologies we have that $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than $\mathbb{R}$.

But we cannot arrive that $T' \subset T''$ and $T'' \subset T'$

Hence $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.

Definition.

A ***subbasis*** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The ***topology generated by the subbasis S*** is defined to be the collection $\mathcal{T}$ of all unions of finite intersection of elements of $\mathcal{S}$.

1.3.The order Topology

Definition.

If X is a simply ordered set, there is a standard topology for X, defined using the order relation. It is called the ***order topology***.

Suppose that X is a set having a simple order relation <. Given elements a and b of X such that $a < b$, there are four subsets of X that are called the ***intervals*** determined by a and b. They are the following:

$$(a, b) = \{x|a < x < b\},$$

$$(a, b] = \{x|a < x \leq b\},$$

$$[a, b) = \{x|a \leq x < b\},$$

$$[a, b] = \{x|a \leq x \leq b\}.$$

A set of the first type is called an ***open interval*** in X, a set of the last type is called a ***closed interval*** in X, and sets of the second and third types are called ***half-open intervals***.

Definition.

Let X be a set with a simple order relation; assume X has more than one element. Let $\mathfrak{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathfrak{B}$ is a basis for a topology on X, which is called the ***order topology***.

If X has no smallest element, there are no sets of type (2), and if X has no largest element, there are no sets of type (3).

Example 1.

The standard topology on R is the order topology derived from the usual order on R**.**

Example 2.

Consider the set $\mathbb{R} \times \mathbb{R}$ in the dictionary order; we shall denote the general element of $\mathbb{R} \times \mathbb{R}$ by $x \times y$. The set $\mathbb{R} \times \mathbb{R}$ has neither a largest nor a smallest

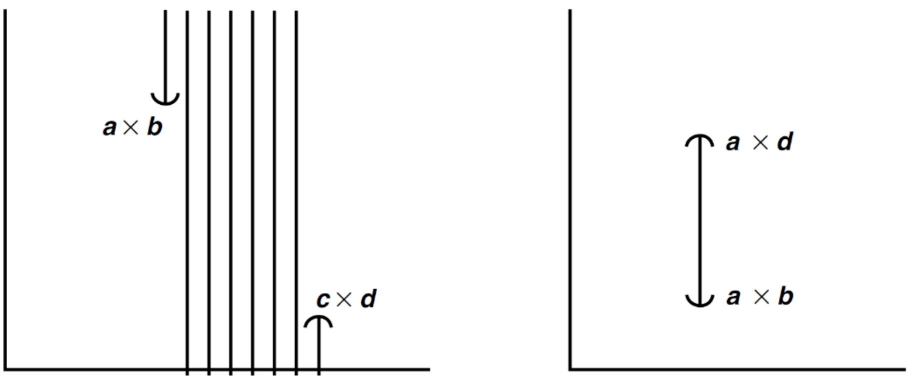


Figure 1.3.1

element, so the order topology on ℝ × ℝ has as basis the collection of all open intervals of the form *(a × b, c × d)* for $a < c$, and for $a = c$ and $b < d$. These two types of intervals are indicated in Figure 1.3.1. The subcollection consisting of only intervals of the second type is also a basis for the order topology on ℝ × ℝ.

Example3.

The positive integers Z_+ form an ordered set with a smallest element. The order topology on Z_+ is the discrete topology, for every one-point set is open: If $n > 1$, then the one-point set $\{n\} = (n - 1, n + 1)$ is a basis element; and if $n = 1$, the one-point set $\{1\} = [1, 2)$ is a basis element.

Example 4.

The set $X = \{1, 2\} \times Z_+$ in the dictionary order is another example of an ordered set with a smallest element. Denoting $1 \times n$ by a_n and $2 \times n$ by b_n, we can represent *X* by

$$a_1 \; , a_2, \ldots ; \; b_1, b_2, \ldots$$

The order topology on *X* is *not* the discrete topology. Most one-point sets are open, but there is an exception–the one-point set $\{b_1\}$. Any open set containing b_1 must contain a basis element about b_1 (by definition), and any basis element containing b_1 contains points of the a_i sequence.

Definition.

If X is an ordered set, and a is an element of X, there are four subsets of X that are called the ***rays*** determined by a. They are the following:

$$(a, +\infty) = \{x \mid x > a\},$$

$$(-\infty, a) = \{x \mid x < a\},$$

$$[a, +\infty) = \{x \mid x \geq a\},$$

$$(-\infty, a] = \{x \mid x \leq a\}.$$

Sets of the first two types are called ***open rays***, and sets of the last two types are called ***closed rays***.

1.4.The Product Topology on $X \times Y$

If X and Y are topological spaces, there is a standard way of defining a topology on
the cartesian product $X \times Y$.

Definition.

Let X and Y be topological spaces. The ***product topology*** on $X \times Y$ is the topology having as basis the collection $\mathfrak{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Note. The collection $\mathfrak{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y. Then $\mathfrak{B}$ is a basis for $X \times Y$ but not a topology on $X \times Y$.

For, the first condition is trivial, since $X \times Y$ is itself a basis element.

Let $U_1 \times V_1, U_2 \times V_2 \in \mathfrak{B}$.

Then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2),$$

Since U_1 *and* U_2 are open in X, $U_1 \cap U_2$ is open in X.

Similarly, $V_1 \cap V_2$ is open in Y.

Therefore, $(U_1 \cap U_2) \times (V_1 \cap V_2)$ is a basis element

$\Rightarrow (U_1 \times V_1) \cap (U_2 \times V_2)$ is a basis element.

∴ the second condition for a basis is satisfied.

Thus $\mathfrak{B}$ is a basis for $X \times Y$. See Figure 1.4.1.

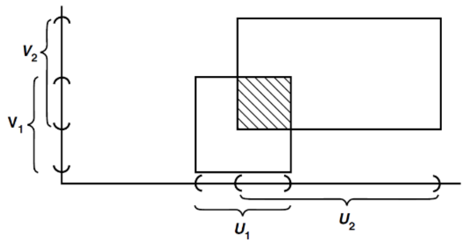


Figure 1.4.1

Note that the collection $\mathfrak{B}$ is not a topology on $X \times Y$. The union of the two rectangles pictured in Figure 1.4.1, for instance, is not a product of two sets, so it cannot belong to $\mathfrak{B}$; however, it is open in $X \times Y$.

Theorem 1.4.1.

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection $\mathcal{D} = \{B \times C \mid B \in \mathcal{B}$ and $C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

Proof.

We apply Lemma 1.2.1,

Given an open set W of $X \times Y$ and a point $x \times y$ of W

Then by definition of the product topology, there is a basis element $U \times V$ such that $x \times y \in U \times V \subset W$.

Because $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively, we can choose an element B of $\mathcal{B}$ such that $x \in B \subset U$, and an element C of $\mathcal{C}$ such that $y \in C \subset V$. Then $x \times y \in B \times C \subset W$.

Thus, the collection $\mathcal{D}$ meets the criterion of Lemma 13.2, so $\mathcal{D}$ is a basis for $X \times Y$.

Example 1.

We have a standard topology on $\mathbb{R}$: the order topology. The product of this topology with itself is called the ***standard topology*** on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. It has as basis the collection of all products of open sets of R, but the theorem 1.4.1 tells us that the much smaller collection of all products $(a, b) \times (c, d)$ of open intervals in R will also serve as a basis for the topology of $\mathbb{R}^2$. Each such set can be pictured as the interior of a rectangle in $\mathbb{R}^2$.

Definition.

Let $\pi_1 : X \times Y \to X$ be defined by the equation $\pi_1(x, y) = x$; let $\pi_2 : X \times Y \to Y$ be defined by the equation $\pi_2(x, y) = y$. The maps π_1 and π_2 are called the ***projections*** of $X \times Y$ onto its first and second factors, respectively.

Remark.

If U is an open subset of X, then the set $\pi_1^{-1}(U) = U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then $\pi_2^{-1}(V) = X \times V$, which is also open in $X \times Y$. The intersection of these two sets is the set $U \times V$, as indicated in Figure 1.4.2.

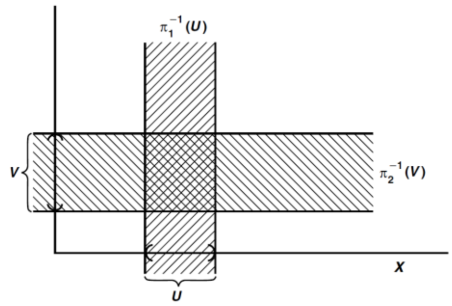


Figure 1.4.2

Theorem 1.4.2.

The collection $\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_1^{-1}(V) \mid V \text{ open in } Y\}$ is a subbasis for the product topology on $X \times Y$.

Proof.

Let $\mathcal{T}$ denote the product topology on $X \times Y$;

Let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$.

Because every element of $\mathcal{S}$ belongs to $\mathcal{T}$, so do arbitrary unions of finite intersections of elements of $\mathcal{S}$.

Thus $\mathcal{T}' \subset \mathcal{T}$.

On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of $\mathcal{S}$, since $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$.

Therefore, $U \times V \in \mathcal{T}$, so that $\mathcal{T} \subset \mathcal{T}'$ as well.

1.5. The Subspace Topology

Definition.

Let *X* be a topological space with topology $\mathcal{T}$. If *Y* is a subset of *X*, the collection $\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$ is a topology on *Y*, called the ***subspace topology***. With this topology, *Y* is called a ***subspace*** of *X*; its open sets consist of all intersections of open sets of *X* with *Y*.

Lemma 1.5.1.

If $\mathcal{B}$ is a basis for the topology of *X* then the collection

$$\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on *Y*.

Proof.

Consider U is open in X.

Given $\mathcal{B}$ is a basis for the topology of X.

We can choose an element B of $\mathcal{B}$ such that $y \in B \subset U$.

Then $y \in B \cap Y \subset U \cap Y$, since $\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$.

It follows from Lemma 1.2.2 that $\mathcal{B}_Y$ is a basis for the subspace topology on Y.

Definition.

If Y is a subspace of X, we say that a set U is ***open** in* Y (or **open *relative*** to Y) if it belongs to the topology of Y; this implies in particular that it is a subset of Y. We say that U is ***open** in* X if it belongs to the topology of X.

Lemma 1.5.2.

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof.

Given U is open in Y and Y is open in X.

Since U is open in Y and Y is a subspace of X then $U = Y \cap V$ where V is open

in X.

Since Y and V are both open in X, $Y \cap V$ is open in X.

Therefore, U is open in X.

Theorem 1.5.3.

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof.

The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y.

Then $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Since $U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product on $A \times B$.

The bases for the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same.

Hence the topologies are the same.

Remark.

Now let *X* be an ordered set in the order topology, and let *Y* be a subset of *X*. The order relation on *X*, when restricted to *Y* , makes *Y* into an ordered set. However, *the resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X*. We give one example where the subspace and order topologies on *Y* agree, and two examples where they do not.

Example 1.

Consider the subset $Y = [0,1]$ of the real line ℝ, in the *subspace* topology.

The subspace topology has as basis all sets of the form $(a, b) \cap Y$, where (a, b) is an open interval in ℝ. Such a set is of one of the following types:

$$(a, b) \cap Y = \begin{cases} (a, b) & if\ a\ and\ b\ are\ in\ Y, \\ [0, b) & if\ only\ b\ is\ in\ Y\ , \\ (a, 1] & if\ only\ a\ is\ in\ Y\ , \\ Y\ or\ \emptyset & if\ neither\ a\ nor\ b\ is\ in\ Y \end{cases}$$

By definition, each of these sets is open in *Y*. But sets of the second and third types are not open in the larger space R.

Note that these sets form a basis for the *order* topology on *Y*. Thus, we see that in the case of the set *Y* = [0, 1], its subspace topology (as a subspace of R) and its order topology are the same.

Example 2.

Let *Y* be the subset $[0,1) \cup \{2\}$ of ℝ. In the subspace topology on *Y* the one-point set {2} is open, because it is the intersection of the open set $\left(\frac{3}{2}, \frac{5}{2}\right)$ with *Y*. But in the order topology on *Y*, the set {2} is not open. Any basis element for the order topology on *Y* that contains 2 is of the form

$$\{x \mid x \in Y\ and\ a < x \leq 2\}$$

for some $a \in Y$; such a set necessarily contains points of *Y* less than 2.

Example 3.

Let $I = [0,1]$. The dictionary order on $I \times I$ is just the restriction to $I \times I$ of the dictionary order on the plane $R \times R$.

However, the dictionary order topology on $I \times I$ is not the same as the subspace topology on $I \times I$ obtained from the dictionary order topology on $R \times R$!.

For example, the set $\{1/2\} \times (1/2,1]$ is open in $I \times I$ in the subspace topology, but not in the order topology. See Figure 1.5.1.

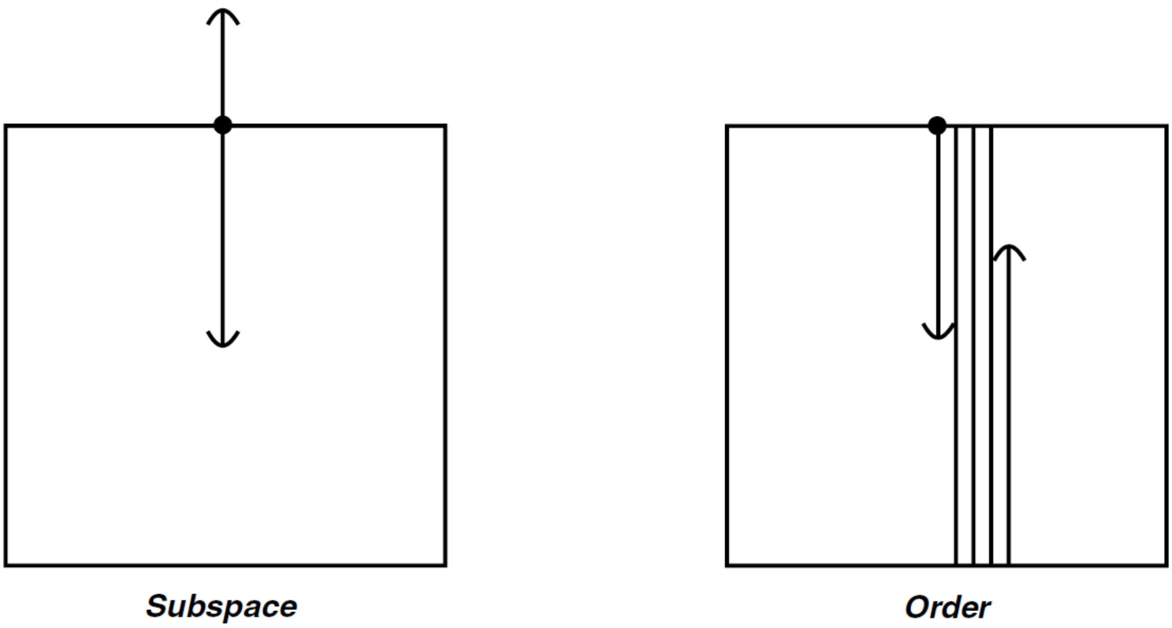


Figure 1.5.1

The set $I \times I$ in the dictionary order topology will be called the *ordered square*, and denoted by I_0^2.

Definition.

Given an ordered set X, let us say that a subset Y of X is ***convex*** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y. Note that intervals and rays in X are convex in X.

Theorem 1.5.4.

Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Proof.

Consider the ray $(a, +\infty)$ in X.

If $a \in Y$, then $(a, +\infty) \cap Y = \{x | x \in Y \; and \; x > a\}$; this is an open ray of the ordered set Y.

If $a \notin Y$, then a is either a lower bound on Y or an upper bound on Y, since Y is convex.

If $a \in Y$, the set $(a, +\infty) \cap Y$ equals all of Y. If $a \notin Y$, it is empty.

Similarly the intersection of the ray $(-\infty, a) \cap Y$ is either an open ray of Y, or Y itself or empty.

Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y and since each is open in the order topology, the order topology contains the subspace topology.

Conversely, Y equals the intersection of X with Y, that is $X \cap Y = Y$.

So, it is open in the subspace topology on Y. The order topology is contained in the subspace topology. Therefore, the order topology and subspace topology are same.

1.6. Closed Sets and Limit Points

Definition.

A subset A of a topological space X is said to be *closed* if the set $X - A$ is open.

Example 1.

(i)The subset $[a, b]$ of $\mathbb{R}$ is closed because its complement

$\mathbb{R} - [a, b] = (-\infty, a) \cup (b, +\infty)$, is open.

(ii)Similarly, $[a, +\infty)$ is closed, because its complement $(-\infty, a)$ is open.

(iii)The subset $[a, b)$ of $\mathbb{R}$ is neither open nor closed.

Example 2.

In the plane $\mathbb{R}^2$, the set $\{x \times y | \, x \geq 0 \; and \; y \geq 0\}$ is closed, because its complement is the union of the two sets $(-\infty, 0) \times \mathbb{R}$ and $\mathbb{R} \times (-\infty, 0)$, each of which is a product of open sets of $\mathbb{R}$ and is, therefore, open in $\mathbb{R}^2$.

Example 3.

In the finite complement topology on a set *X*, the closed sets consist of *X* itself and all finite subsets of *X*.

Example 4.

In the discrete topology on the set *X*, every set is open; it follows that every set is closed as well.

Example 5.

Consider the following subset of the real line:

$$Y = [0,1] \cup (2,3),$$

in the subspace topology. In this space, the set $[0,1]$ is open, since it is the intersection of the open set $(-\frac{1}{2}, \frac{3}{2})$ of $\mathbb{R}$ with Y. Similarly, $(2, 3)$ is open as a subset of Y; it is even open as a subset of $\mathbb{R}$. Since $[0,1]$ and $(2,3)$ are complements in Y of each other, we conclude that both $[0,1]$ and $(2,3)$ are closed as subsets of Y.

Theorem 1.6.1.

Let X be a topological space. Then the following conditions hold:

(1) $\emptyset$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof.

(1) $\emptyset$ and X are closed because they are the complements of the open set X and $\emptyset$ respectively.

(2) Consider a collection of closed sets $\{A_\alpha\}_{\alpha\in J}$, we apply De Morgan's law,

$$X - \bigcap_{\alpha\in J} A_\alpha = \bigcup_{\alpha\in J} (X - A_\alpha)$$

Since the sets $X - A_\alpha$ are open. By definition of closed sets, the right side of this equation represents an arbitrary union of open sets and is thus open.

Therefore, $\bigcap_{\alpha\in J} A_\alpha$ is closed.

(3) Similarly, if A_i is closed for $i = 1, 2, \cdots, n$. Consider the equation

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$$

The set on the right side of this equation is a finite intersection of open sets and is therefore open. Hence $\bigcup_{i=1}^{n} A_i$ is closed.

Definition.

If Y is a subspace of X, we say that a set A is ***closed** in* Y if A is a subset of Y and if A is closed in the subspace topology of Y (that is, if $Y - A$ is open in Y).

Theorem 1.6.2.

Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

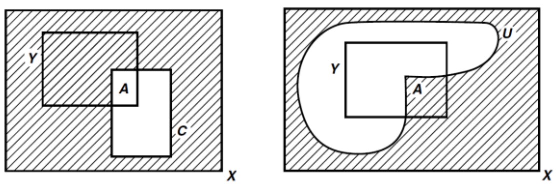
Proof.

Assume that $A = C \cap Y$, where C is closed in X. See Figure 1.6.1.

Then $X - C$ is open in X, so that $(X - C) \cap Y$ is open in Y.

By the definition of the subspace topology, but $(X - C) \cap Y = Y - A$.

Hence $Y - A$ is open inY, so that A is closed in Y.



Conversely, assume that A is closed in Y. See Figure 1.6.1

Then $Y - A$ is open in Y.

By definition, it equals the intersection of an open set U of X with Y.

The set $X - U$ is closed in X and $A = Y \cap (X - U)$.

Hence A equals the intersection of a closed set of X with Y.

Theorem 1.6.3.

Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Proof.

Given A is closed in Y and Y is closed in X.

Since A is closed in Y and Y is a subspace of X.

Let $A = Y \cap (X - B)$ where $X - B$ is open in X. Then B is closed in X.

Since Y and B are both closed in X. $Then\ Y \cap (X - B)$ is closed in X. Therefore, A is closed in X.

Closure and Interior of a Set

Definition.

Given a subset A of a topological space X, the ***interior*** of A is defined as the union of all open sets contained in A, and the ***closure*** of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by $Int\ A$ and the closure of A is denoted by $Cl\ A$ or by $\bar{A}$. Obviously Int A is an open set and A is a closed set; furthermore,

$$Int\ A \subset A \subset \bar{A}.$$

If A is open, $A = Int\ A$; while if A is closed, $A = \bar{A}$.

Theorem 1.6.4.

Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Proof.

Let B denote the closure of A in Y. The set A is closed in X, so A ∩ Y is closed in Y.

By Theorem 1.6.4, since $\bar{A} \cap Y$ contains A and since B is closed.

By definition B equals the intersection of all closed subsets of Y containing A, we must have $B \cap (\bar{A} \cap Y)$.

On the other hand, we know that B is closed in Y. By Theorem 1.6.4, $B = C \cap Y$ for some set C closed in X.

Then C is a closed set of X containing A; because A is the intersection of all such closed sets, we conclude that A ⊂ C. Then

$(A \cap Y) \subset (C \cap Y) = B$. Therefore, $B = \bar{A} \cap Y$.

Note. We shall say that a set A ***intersects*** a set B if the intersection $A \cap B$ is not empty.

Theorem 1.6.5.

Let A be a subset of the topological space X.

(a) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

Proof. (a)We prove this theorem by contrapositive method.

If x is not in A, since A is closed, A = A. The set U = X − A is an open set

containing x that does not intersect A.

Conversely, if there exists an open set U containing x which does not intersect

A. Then X − U is a closed set containing A.

By definition of the closure A, the set X − U must contain A, since x ∈ U.

Therefore, x cannot be in A.

(b) Write the definition of topology generated by basis,if every open set x intersects

A, so does every basis element B containing x, because B is an open set.

Conversely, if every basis element containing x intersects A, so does every open

set U containing x, because U contains a basis element that contains x.

Definition.

If A is a subset of the topological space X and if x is a point of X, we say that x is a ***limit point***(or "***cluster point***" or "***point of accumulation***") of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of A − {x}. The point x may lie in A or not; for this definition it does not matter.

Theorem 1.6.6.

Let A be a subset of the topological space X; let A′ be the set of all limit points of A. Then A = $\bar{A}$ ∪ A′.

Proof.

Let A′ be the set of all limit points of A.

If x ∈ A', every neighborhood of x intersects of A in a point different from x. By

Theorem 1.6.5, x ∈ A. Then $A' \subset \bar{A}$.

By definition of closure, $A \subset \bar{A}$. Therefore, $A \cup A' \subset \bar{A}$.

Conversely, let x ∈ $\bar{A}$

To show that $\bar{A}$ ⊂ A ∪ A′

If x ∈ A then it is trivially true for x ∈ A ∪ A′

Suppose $x \notin A$. Since x ∈ $\bar{A}$, by 0.6.8, we know that every neighborhood U of x

intersect A, because x ∉ A, the set U must intersect A in a point different from

x. Then x ∈ A′ so that x ∈ A ∪ A′

Then $\bar{A}$ ⊂ A ∪ A′

Therefore, A = A ∪ A′

Corollary 1.6.7.

A subset of a topological space is closed if and only if it contains all its limit points.

Proof.

The set A is closed iff $A = \bar{A}$. By Theorem 1.6.7, $A' \subset A$.

Definition.

A topological space X is called a ***Hausdroff space*** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint.

Theorem 1.6.8.

Every finite point set in a Hausdorff space X is closed.

Proof.

It is enough to show that every one-point set $\{x_0\}$ is closed.

If x is a point of X different from x_0, then x and $x_0$have disjoint neighborhoods

U and V respectively.

Since U does not intersect $\{x_0\}$, the point x cannot belong to the closure of the

set $\{x_0\}$.

As a result, the closure of the set $\{x_0\}$ is $\{x_0\}$ itself.

Therefore, $\{x_0\}$ is closed.

Note: The condition that finite point sets be closed is in fact weaker than the Hausdroff condition. For example, the real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed. The condition that finite point sets be closed has been given a name of its own; it is called the T_1 *axiom*.

Theorem 1.6.9.

Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof.

If every neighborhood of x intersects A in infinitely many points, it certainly intersects A in some point other than x itself, so that x is a limit point of A.

Conversely, suppose that x is a limit point of A and suppose some neighborhood U of x intersects A in only finitely many points.

Let $\{x_1, x_2, \cdots, x_m\}$ be the points of $U \cap (A - \{x\})$.

The set $X - \{x_1, x_2, \cdots, x_m\}$ is an open set of X, since the finite point set $\{x_1, x_2, \cdots, x_m\}$ is closed then

$$U \cap (X - \{x_1, x_2, \cdots, x_m\})$$

is a neighborhood of x that does not intersects the set $A - \{x\}$.

Since $\{x_1, x_2, \cdots, x_m\}$be points of $U \cap (A - \{x\})$.

This contradicts the assumption that x is a limit point of A.

Theorem 1.6.10.

If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Proof.

Suppose that x_n is a sequence of points of X that converges to x.

If $y \neq x$, let U and V be disjoint neighborhoods of x and y respectively.

Since U contains x_n for all but finitely many values of n, the set V cannot contains x_n.

Therefore, $x_n$cannot converge.

If the sequence $x_n$of points of the Hausdorff space X converges to the point x of

X, we often write $x_n \rightarrow x$.

Therefore, x is the limit of the sequence x_n.

Theorem 1.6.11.

Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.

Proof.

Let X and Y be two Hausdorff spaces.

To prove $X \times Y$ is Hausdorff.

Let $x_1 \times y_1$ and $x_2 \times y_2$ be two distinct points of $X \times Y$.

Then x_1, x_2 are distinct points of X and X is a Hausdorff space, there exists neighborhood U_1 and U_2 of x_1 and x_2 such that $U_1 \cap U_2 = \emptyset$

Similarly, y_1, y_2 are distinct point of Y and Y is a Hausdorff space, there exists neighborhood V_1 and V_2 of y_1 and y_2 such that $V_1 \cap V_2 = \emptyset$.

Then clearly $U_1 \times V_1$ and $U_2 \times V_2$ are open sets in $X \times Y$ containing $x_1 \times y_1$ and

$x_2 \times y_2$ such that $(U_1 \times V_1) \cap (U_2 \times V_2) = \emptyset$.

Therefore, $X \times Y$ is a Hausdorff space.

Let X be a Hausdorff space and let Y be a subspace.

To prove Y is a Hausdorff space.

Let y_1, y_2 be two distinct points of Y and Y containing X. Then y_1 and y_2 are distinct points in X and X is Hausdorff there exists neighborhood U_1 and U_2 of y_1 and y_2 such that $U_1 \cap U_2 = \emptyset$. Then $U_1 \cap Y$ and $U_2 \cap Y$ are distinct neighborhoods of y_1 and y_2 in Y .

Therefore, Y is a Hausdorff space.

UNIT – 2

CONTINUOUS FUNCTIONS

2.1. Continuous Functions

Continuity of a function.

Definition.

Let X and Y be a topological spaces. A function $f: X \to Y$ is said to be ***continuous*** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Note. $f^{-1}(V)$ is the set of all points x of X for which $f(x) \in V$; it is empty if V does not intersect the image set *f(X)* of f.

Remark.

If the topology of the range space Y is given by a basis $\mathfrak{B}$, then to prove the continuity of f, it is sufficient to prove that the inverse of every basis element is open:

For, the arbitrary open set V of Y can be written as $V = \bigcup_{\alpha \in J} B_\alpha$ of basis element

Then $f^{-1}(V) = f^{-1}\left(\bigcup_{\alpha \in J} B_\alpha\right) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)$ is open.

Remark.

If the topology of Y is given by a subbasis S_1 then to prove the continuity of f, it is sufficient to prove that the inverse images of each subbasis element is open:

For, the arbitrary basis element B of Y can be written as the finite intersection of subbasis element.

$$B = S_1 \cap S_2 \cap \dots \dots \cap S_n$$

$$\Rightarrow f^{-1}(B) = f^{-1}(S_1) \cap f^{-1}(S_2) \cap \dots \dots \cap f^{-1}(S_n)$$

$\therefore f^{-1}(B)$ is open if each set $f^{-1}(S_1), f^{-1}(S_2) \dots \dots f^{-1}(S_n)$ is open.

Example 1.

Prove that our definition of continuity implies $\epsilon - \delta$ definition

Solution.

Consider $f: \mathbb{R} \to \mathbb{R}$ is a real valued function of a real variables

Let $x_0 \epsilon \mathbb{R}$ (domain)

Let $\epsilon > 0$ be given and $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$

Then V is an open set of the range space $\mathbb{R}$.

By definition of continuity, $f^{-1}(V)$ is open set in the domain space $\mathbb{R}$.

Since $f(x_0) \in f^{-1}(V)$

$\therefore$ We can choose an open interval (a, b)

Such that $x_0 \in (a, b) \subseteq f^{-1}(V)$

Let $\delta = \min\{x_0 - a, b - x_0\}$

Then $\delta > 0$

Let $|x - x_0| < \delta \implies x \in (a, b)$

$\implies x \in f^{-1}(V)$

$\implies f(x) \in V$

$\implies f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$

$\implies f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$

$\implies -\epsilon < f(x) - f(x_0) < \epsilon$

$\implies |f(x) - f(x_0)| < \epsilon$

i.e., $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$

Example 2.

Let $\mathbb{R}$ denote the set of real numbers n its usual topology and let $\mathbb{R}_l$ denote the same set with lower limit topology.

Let $f: \mathbb{R} \to \mathbb{R}_l$ be the identity function.

Such that, $f(x) = x$, then f is not continuous

For, the inverse image of $[a, b)$

i.e., $f^{-1}[(a, b)] = [a, b)$ of equals itself.

But this interval is not open in $\mathbb{R}$, on the other hand the identity function $g: \mathbb{R}_l$ ->$\mathbb{R}$ is continuous.

Since, inverse image of $[a, b)$ is itself open in $\mathbb{R}_l$.

Theorem 2.1.1.

Let X and Y be the topological spaces. Let $f: X \to Y$ be a mapping. Then the following are equivalent.

(i) f is continuous.

(ii) $for\ every\ subset\ A\ of\ X\ one\ has\ f(\bar{A}) \subset \overline{f(A)}$.

(iii) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(iv) For each $x \in X$ and each neighbourhood V of $f(x)$, there is neighbourhood U $of\ x$ such that $f(U) \subset V$.

If the condition (iv) holds for the point $x \in X$. We say that f is continuous at the point x.

Proof.

To show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$ and $(i) \Rightarrow (iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$

Suppose that f is continuous

To prove, $f(\bar{A}) \subset \overline{f(A)}$

Let $x \in \bar{A}$, then $f(x) \in f(\bar{A})$

Claim: $f(x) \in \overline{f(A)}$

Since f is continuous, $f^{-1}(V)$ is an open set of X containing x, where V be a neighborhood of $f(x)$.

Now, $f^{-1}(V)$ intersects A in some point y

$\Longrightarrow V$ intersect $f(A)$ in the point $f(y)$

$\Longrightarrow f(x) \in \overline{f(A)}$

$\therefore f(\bar{A}) \subseteq \overline{f(A)}$.

To prove $(ii) \Rightarrow (iii)$

Let B be closed in Y. Let $A = f^{-1}(B)$.

To prove $f^{-1}(B)$ is closed in X

Let $A = f^{-1}(B)$

i.e., To prove A is closed in X

It is enough to prove $A = \bar{A}$

Always, $A \subseteq \bar{A}$

By elementary set theory, we have $f(A) = f(f^{-1}(B)) \subset B$

If $x \in \bar{A}$, then $f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$.

Since A ⊂ A, therefore, $\bar{A}$ = A.

To prove, (iii) ⟹ (i)

Let V be an open set in Y. The set $B = Y - V$.

Then $f^{-1}(B) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$

Now B is a closed set of Y then $f^{-1}(B)$ is closed in X(By hypothesis).

Then $f^{-1}(V)$ is open in X.

Therefore, f is continuous.

To prove (i) $\Longrightarrow (iv)$

Let $x \in X$ and let V be a neighbourhood of $f(x)$

Then the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(V) \subset V$

To prove (iv) $\Longrightarrow$(i)

Let V be an open set of Y and Let $x \in f^{-1}(V)$

Then $f(x) \in V$

By our hypothesis there is a neighbourhood U_x of x such that $f(U_x) \subset V$

Then $U_x \subset f^{-1}(V)$

It follows that $f^{-1}(V)$ can be written as the union of open set U_x

$\therefore f^{-1}(V)$ is open in X

Hence f is continuous.

Homeomorphism

Let X and Y be a topological spaces. Let $f: X \to Y$ be a bijection. If both the function f and inverse function $f^{-1}: Y \to X$ are continuous. Then f is called a ***Homeomorphism***.

Remark 1.

The condition that f^{-1} is continuous says that for each open set U of X, the inverse image of U under the map $f^{-1}: V \to X$ is open in Y.

But $(f^{-1})^{-1} = f$

$\therefore (f^{-1})^{-1}(V) = f(V)$

Hence a homeomorphism is a bijective correspondence $f: X \to Y$ such that $f(V)$ is open iff U is open.

Remark 2

The above remark shows that the homeomorphism $f: X \to Y$ gives as a bijective correspondence not only between X and Y but between the collection of open sets of X and Y

As a result, any property of X that is entirely expressed in terms of the topology of X

Yields, via the correspondence f the corresponding property for the space Y, such a property of X is called a topology property of X.

Topological Imbedding

Suppose that $f: X \to Y$ is an injective continuous map where X are Y are topological space.

Let Z be the image of $f(X)$ condered as a subspace of Y

Then the function $f': X \to Z$. Obtained by restricting the range of f is bijective.

If f' happens to be a homeomorphism of X with Z, we say that the map $f: X \to Y$ is a topological imbedding (or) simply on imbedding of X in Y.

Remark.

Let $f: A \to B$. If there are function, $g: B \to A$ and $h: B \to A$ show that $g[f(x)] = a, \forall a \in A$ and $f[h(b)] = b, \forall b \in B$, then f is bijective and $g = b = f^{-1}$.

Example 1.

Show that the function $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x + 1$ is a homeomorphism.

Solution.

Define $g: \mathbb{R} \to \mathbb{R}$ by $g(y) = \frac{y-1}{3}$

Now, $g[f(x)] = \frac{f(x)-1}{3}$

$= \frac{3x+1-1}{3}$

$= \frac{3x}{3}$

$g[f(x)] = x$

and $f[g(y)] = 3[g(y)] + 1$

$= 3\left(\frac{y-1}{3}\right) + 1$

$= y - 1 + 1$

$f[g(y)] = y$

∴ By the above result, f is bijective and $g = f^{-1}$ we know that the algebraic functions are continuous.

Since f and f^{-1} are algebraic functions, we have f and f^{-1} are continuous.

Hence f is a homeomorphism

Example 2.

Show that the function $F: (-1,1) \to R$ defined by $F(x) = \frac{x}{1-x^2}$ $is\ a$ homeomorphism.

Solution.

Let $y = \frac{x}{1-x^2}$

$y(1-x^2) = x$

$y - x^2 y = x$

$x^2 y + x - y = 0$

$\therefore x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Now, $x = \frac{-1+\sqrt{1+4y^2}}{2y} \times \frac{-1-\sqrt{1+4y^2}}{-1-\sqrt{1+4y^2}}$

$$= \frac{1-(1+4y^2)}{-2y\left(-1+\sqrt{1+4y^2}\right)}$$

$$= \frac{-4y^2}{-2y(+1+\sqrt{1+4y^2})}$$

$$= \frac{2y}{1+\sqrt{1+4y^2}}$$

Define $G \to R \to (-1,1)$ by $G(y) = \frac{2y}{1-\sqrt{1+4y^2}}$

$$G[f(x)] = \frac{2[F(x)]}{1+\sqrt{1+4(F(x))^2}}$$

$$= \frac{2\left(\frac{x}{1-x^2}\right)}{1+\sqrt{1+4\left(\frac{x}{1-x^2}\right)^2}}$$

$$= \frac{2x/(1-x^2)}{1+\frac{\sqrt{(1-x^2)^2+4x^2}}{(1-x^2)^2}}$$

$$= \frac{2x/(1-x^2)}{1+\frac{\sqrt{(1-x^2)^2+4x^2}}{(1-x^2)}}$$

$$= \frac{2x/(1-x^2)}{\frac{(1-x^2)\sqrt{1+x^4-2x^2+4x^2}}{(1-x^2)}}$$

$$= \frac{2x}{(1-x^2)+\sqrt{1+x^4+2x^2}}$$

$$= \frac{2x}{(1-x^2)+\sqrt{(1+x^2)^2}}$$

$$= \frac{2x}{(1-x^2)+(1+x^2)}$$

$$= \frac{2x}{2}$$

$$G[f(x)] = x$$

$$f[g(y)] = \frac{g(y)}{1-(g(y))^2}$$

$$= \frac{2y/1+\sqrt{1+4y^2}}{1-4y^2/(1+\sqrt{1+4y^2})^2}$$

$$= \frac{2y/1+\sqrt{1+4y^2}}{\left(\sqrt{1+4y^2}\right)^2-4y^2/\,(1+\sqrt{1+4y^2})^2}$$

$$= \frac{2y}{\frac{\left(1+\sqrt{1+4y^2}\right)^2-4y^2}{\left(1+\sqrt{1+4y^2}\right)}}$$

$$= \frac{2y\left(1+\sqrt{1+4y^2}\right)}{1+1+4y^2+2\left(\sqrt{1+4y^2}\right)-4y^2}$$

$$= \frac{2y\left(1+\sqrt{1+4y^2}\right)}{2\left(1+\sqrt{1+4y^2}\right)}$$

$f[g(y)] = y$

By the above result F is bijective and $G = F^{-1}$

We know than the algebraic function and square root functions are continuous

Since F and F^{-1} are algebraic and square root function

We have F and F^{-1} are continuous

Hence F is homeomorphism

Example 3.

The identity function $g: \mathbb{R}_l \to \mathbb{R}$ is not a homeomorphism.

For , since g is bijective and g is continuous but g^{-1} is not a continuous junction.

Example 4.

Let S′ denote the unit circle.

$S' = \{x \times y/x^2 + y^2 = 1\}$ considered as a subspace of the plane $\mathbb{R}^3$

Let $f: [0,1) \to S'$ be the map defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$

Then f is not a homeomorphism.

Since, $\cos 2\pi t, \sin 2\pi t$ are continuous functions.

Then clearly f is bijective and continuous functions.

Let $U = \left[0, \frac{1}{4}\right)$ in $[0,1)$

Then the image of U is not open in S' for the point $p = f(U)$ lies in no open set V of $\mathbb{R}^2$

Such that $V \cap S' \subset f(U)$

$\therefore f^{-1}$ is not continuous.

Hence, f is not a homeomorphism.

Constructing Continuous Functions

Theorem 2.1.2. (Rules for constructing continuous functions)

Let X, Y and Z a topological spaces.

a) (constant function) If $f: X \to Y$ maps all of X into the single point x_0 of Y, then f is continuous
b) (Inclusion) If A is a subspace of X, the inclusion function $j: A \to X$ is continuous.
c) (composition) If $f: X \to Y$ and $g: Y \to Z$ are continuous, then the map $g_o f: X \to Z$ is continuous.
d) (Restricting the domain) If $f: X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f|A: A \to Y$ is continuous.
e) (Restricting (or) expanding the range)

    Let $f: X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(x)$, then the function $g: X \to Z$ obtained by restricting the range of f is continuous. If Z is space having Y as subspace. Then the function $h: X \to Z$ obtained by expanding the range of f is continuous.

f) (Local formulation of continuity)

The map $f: X \to Y$ is continuous if X can be written as the union of open sets. $U_\propto$. Such that $f|U_\propto$ is continuous for each $\propto$.

e) (Continuity at each point)

The map $f: X \to Y$ is continuous if for each $x \in X$ and each neighbour hood V of $f(x)$, there is a nbd U of x such that $f(U) \subset V$

[Note: If the condition (g) holds for a particular point x of X, we say that f is continuous at the point x]

Proof.

a) Let $f(x) = y_0 \ \forall x \in X$

Let V be an open set in Y

Then $f^{-1}(V) = \begin{cases} X & if\ y_0 \in V \\ \emptyset & if\ y_0 \notin V \end{cases}$

In either case, $f^{-1}(V)$ is open

Hence f is continuous.

b) Let U be an open set in Y

Then $f^{-1}(V) = U \cap A$, which is open in A by the definition of subspace of topology.

$\therefore j$ is continuous.

c) Let U be an open set in Z

Since $g: Y \to Z$ is continuous.

$g^{-1}(U)$ is open in Y

Since $f: X \to Y$ is continuous, $f^{-1}\left(g^{-1}(U)\right)$ is open in X

i.e., $(f^{-1} \circ g^{-1})(U)$ is open in Y

but $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$

$\therefore (g \circ f)^{-1}(U)$ is open in Y.

Hence $g \circ f$ is continuous.

d) Here $f|A = f \circ j$, where $j: A \to X$ is the inclusion

Since f and j are continuous, $f \circ j$ is also continuous.

i.e,. $f|A: A \to Y$ is continuous.

e) Let $f: X \to Y$ be continuous

i) If $f(x) \subset Z \subset Y$

To prove $g: X \to Z$ obtained from f is continuous

Let B be open in Z

Since Z is subspace of Y, $B = U \cap Z$, for some open set U in Y

Since $f(x) \subset Z, g^{-1}(B) = \delta^{-1}(V)$ by elementary set theory

Since f is continuous and U is open in $Y, f^{-1}(V)$ is open in X.

$\Rightarrow g^{-1}(B)$ is open in X

g is continuous.

ii) Given $Y \subset Z$

To prove, $h: X \to Z$ is continuous

Here $h = j \circ f$ where $j: Y \to Z$ is the inclusion function

$\Rightarrow h$ is composition of two continuous function

$\Rightarrow h$ is continuous (by(i))

f) If X can be written as the union of open sets $U_\propto$ such that $f|_{U_\propto}$ is continuous for each α.

To prove $f: X \to Y$ is continuous

Let V be an open set in Y

Claim: $f^{-1}(V)$ is open in X

Now, $f^{-1}(V) \cap U_\propto = \{x/x \in f^{-1}(V) \cap U_\propto\}$

$$= \{x/x \in f^{-1}(V) and\ x \in U_\propto\}$$

$$= \left\{\frac{x}{f(x)} \in V\ and\ x \in U_\propto\right\} \quad \to (1)$$

Also, $\left(f|_{U_\propto}\right)^{-1}(V) = \{x \in U_\propto / f(x) \in V\} \quad \to (2)$

From (1) and (2) $F^{-1}(V) \cap U_\propto = \left(f|_{U_\propto}\right)^{-1}(V)$

Since $f|_{U_\propto}: U_\propto \to Y$ continuous and V is open in Y

$\left(f|_{U_\propto}\right)^{-1}(V)\ B$ open in $U_\propto$

But $U_\propto$ is open in X.

$\therefore \left(f|_{U_\propto}\right)^{-1}(V)$ is open in X

$\Rightarrow f^{-1}(V) \cap U_\propto$ is open in X

But $\cup\, [f^{-1}(V) \cap U_\propto] = f^{-1}(V) \cap [U_\propto U_\propto]$

$$= f^{-1}(V) \cap X$$

$$= f^{-1}(V)$$

i.e., $f^{-1}(V)$ is the union of open sets of X

$\Rightarrow f^{-1}(V)$ is open in X

$\therefore f$ is continuous

g) To prove $f: X \to Y$ is continuous

Let V be an open in Y

Claim : $f^{-1}(V)$ is open in X

Let $x \subset f^{-1}(V)$

$\Rightarrow f(x) \in V$

By hypothesis, there is a neighbour hood U_x of x of X

Such that, $f(U_x) \subset V$

Then $U_x \subset f^{-1}(V)$

$\therefore$ for each $x \in f^{-1}(V)$, we can change a neighbour hood U_x of x of $U_x \subset f^{-1}(V)$

$\therefore f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$

$\Rightarrow f^{-1}(V)$ is a union of open sets of X.

$\Rightarrow f^{-1}(V)$ is open in X

Hence f is continuous.

Theorem 2.1.3 (Pasting Lemma)

Let $X = A \cup B$, where A and B are closed in X. Let $f: A \to Y$ and $g: B \to Y$ be continuous if $f(x) = g(x)$ for every $x \in A \cap B$ then f and g combined to give a continuous function $h: X \to Y$ defined by setting $h(x) = g(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Proof.

Let $X = A \cup B$ where A and B are closed in X.

Since $f : A \to Y$ is continuous, $f^{-1}(C)$ is closed in A, where C is closed in Y.

Since $g : B \to Y$ is continuous, $g^{-1}(C)$ is closed in B where C is closed in Y.

If $x \in A,\ h(x) = f(x)$ and if $x \in B, h(x) = g(x)$.

If $x \in A \cup B, h(x) = f(x) \cup g(x)$.

Now $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$.

Then $h^{-1}(C)$ is closed in $A \cup B$.

Then $h^{-1}(C)$ is closed in X.

Therefore, h is continuous.

Theorem 2.1.4 [maps into product]

Let $\mathrm{f}: \mathrm{A} \to \mathrm{X} \times \mathrm{Y}$ be given by the equation $\mathrm{f(a)} = \big(\mathrm{f_1(a)}, \mathrm{f_2(a)}\big)$ then f is conditions iff the function $\mathrm{f_1}: \mathrm{A} \to \mathrm{X}$ and $\mathrm{f_2}: \mathrm{A} \to \mathrm{Y}$ are continuous.

The maps $\mathrm{f_1}$ and $\mathrm{f_2}$ are called the ***co-ordinate functions*** of f.

Proof

Let $\pi_1: X \times Y \to X$ and $\pi_2: X \times Y \to Y$ be projections on to the first and second factors, respectively.

Claim: π_1 and π_2 are continuous

We know that $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$

If U and V are open, these sets are open.

Since $f : A \to X \times Y, \pi_1 : X \times Y \to X\ and\ \pi_2 : X \times Y \to Y$, for every $a \in A$.

Since $f_1 : A \to X$ and $f_2 : A \to Y$

$f_1(a) = \pi_1(f(a))$ and $f_2(a) = \pi_2(f(a))$

If the function f is continuous, then f_1 and f_2 are composites of continuous functions,

f_1 and f_2 are continuous.

Conversely, suppose that f_1 and f_2 are continuous

To prove $f: A \to X \times Y$ is continuous

Let $U \times V$ be any basis element for the product topological space $X \times Y$

Then U and V are open in X and Y respectively

To prove $f^{-1}(U \times V)$ is open in A

Claim, $f^{-1}(U \times V) = f^{-1}(U) \cap f^{-1}(V)$

$\therefore a \in f^{-1}(U \times V) \Leftrightarrow f(a) \in U \times V$

$\Leftrightarrow \big(f_1(a), f_2(a)\big) \in U \times V$

$\Leftrightarrow f_1(a) \in U$ and $f_2(a) \in V$

$\Leftrightarrow a \in f_1^{-1}(U)$ and $a \in f_2^{-1}(V)$

$\Leftrightarrow a \in f_1^{-1}(U) \cap f_2^{-1}(V)$

Since, $f_1: A \to X$ is continuous and U is open in X

We have $f_1^{-1}(U)$ is open in A

Also, since $\boldsymbol{f_2}: A \to Y$ is continuous and V is open in Y

We have $f_2^{-1}(V)$ is open in A

$\therefore f_1^{-1}(U) \cap f_2^{-1}(V)$ is open in A

$\Rightarrow f^{-1}(U \times V)$ is open in A

Hence f is continuous

2.2. The product Topology

Definition.

Let J be an index set. Give a set X. We define a $\boldsymbol{J-tuple}$ of elements of X to be a function $x: J \to X$. If X is an element of J, we often denote the value of x at α by

x_α rather than $x(\alpha)$. We call it the α^{th} co-ordinate of x and we often denote the function x itself by the symbol $(x_\alpha)_{\alpha \in J}$ which is as close as we can come to a tuple notation form arbitrary index set J. We denote the set of all J −tuples of the element of X by X^J.

Definition.

Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of set. Let $X = \bigcup_{\alpha \in J} A_\alpha$. The ***Cartesian product*** of this indexed family, denoted by $\prod_{\alpha \in J} A_\alpha$, is defined to be the set of all J −tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$.

i.e., it is the set of all functions $x: J \to \bigcup_{\alpha \in J} A_\alpha$ such that $X(\alpha) \in A_\alpha$ for each $\alpha \in J$

Definition.

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$ of the collection of all the sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$. The topology generated by this basis is called the ***box topology***.

Remark: The collection $\prod_{\alpha \in J} U_\alpha$ is a basis for a topology on $\prod_{\alpha \in J} X_\alpha$

This collection satisfies the first condition for a basis because πX_α is itself a basis element and it satisfies the second condition because the intersection of any two basis element is another basis element.

$$\left(\prod_{\alpha \in J} U_\alpha\right) \cap \left(\prod_{\alpha \in J} V_\alpha\right) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha)$$

Hence, the above collection is basis for the topology.

Definition.

The function $\pi_\beta: \prod_{\alpha \in J} X_\alpha \to X_\beta$ defined by $\pi_\beta((x_\alpha)_{\alpha\in J}) = x_\beta$ is called the ***projection mapping*** associated with the index β.

Definition.

Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) / U_\beta \text{ is open in } X_\beta\}$$

and let $\mathcal{S}$ denote the union of these collection,

$$\mathcal{S} = U_{\beta \in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called ***product topology***. In this topology $\prod_{\alpha\in J} X_\alpha$ is called the ***product space.***

Theorem 2.2.1[Comparison of the box and product topology]

The box topology on $\prod X_\alpha$ has a basis all set of the form $\prod U_\alpha$ where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has a basis all sets of the form $\prod U_\alpha$. Where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Proof.

By definition of box topology, the basis for box topology on $\prod X_\alpha$ is $\mathcal{B}_b = \{\prod U_\alpha \,|U_\alpha \; is\ open\ in\ X_\alpha\}$.

By definition of product topology, the basis for the topology on $\prod X_\alpha$ is $\mathcal{B}_P$ then $\mathcal{B}_P$ is the collection of all finite intersection of elements of $\mathcal{S}$ and $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta)/U_\beta$ is open in $X_\beta\}$.

Case (i): We take finite intersection of elements of $\mathcal{S}_\beta$.

Let $\pi_\beta^{-1}(U_\beta), \pi_\beta^{-1}(V_\beta), \pi_\beta^{-1}(W_\beta) \in \mathcal{S}_\beta$

Let $B = \pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) \cap \pi_\beta^{-1}(W_\beta)$

$= \pi_\beta^{-1}(U_\beta \cap V_\beta \cap W_\beta) \in \mathcal{S}_\beta \subset \mathfrak{B}_p$

$= \pi_\beta^{-1}(U'_\beta)$, where $U'_\beta = U_\beta \cap V_\beta \cap W_\beta$

$B = \prod_{\alpha \in J} U'_\alpha$ where U'_α is open in X_α for $\alpha = \alpha_1, \alpha_2, \dots \dots \alpha_n$ and $U'_\alpha - X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2, \dots \dots \alpha_n$

Case (ii): We take intersection of elements from different $\mathcal{S}_\beta{}'s$.

Let $B' = \pi_\beta^{-1}(U_{\beta_1}) \cap \pi_\beta^{-1}(U_{\beta_2}) \cap \dots . . \cap \pi_\beta^{-1}(U_{\beta_n})$

$B' = \pi_\beta^{-1}(U_{\beta_1} \cap U_{\beta_2} \cap \dots \dots \cap U_{\beta_n})$

Let $x = (x_\alpha)_{\alpha \in J} \in B'$

Then $x = (x_\alpha)_{\alpha \in J} \in B' \Leftrightarrow (x_\alpha)_{\alpha \in J} \in \pi_\beta^{-1}(U_{\beta_1}) \dots \cap \pi_\beta^{-1}(U_{\beta_n})$

$\Leftrightarrow (x_\alpha)_{\alpha \in J} \in U_{\beta_1} \times \dots . . \times U_{\beta_2} \times \dots . . \times U_{\beta_n} \times \dots \dots$

$\Leftrightarrow x_\alpha \in U_\alpha$ for $\alpha = \beta_1, \beta_2, \beta_3, \dots \dots \beta_n$ and $x_\alpha \in X_\alpha$ for

$\alpha = \beta_1, \beta_2, \beta_3, \dots \dots \beta_n$

$\Leftrightarrow (x_\alpha) \in \prod_{\alpha \in J} U_\alpha$ where is open in X_α

For $\alpha = \beta_1, \beta_2, \beta_3, \dots \dots \beta_n$ and $U_\alpha = X_\alpha$ for $\alpha \neq \beta_1, \beta_2, \beta_3, \dots \dots \beta_n$

$B' = \prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α

Hence in both cases we get every basis elements of the product topology in $\prod \quad X_\alpha$ is of the form $\prod \quad U_\alpha$.

Where U_α is open in X_α and $U_n = X_\alpha$ except for finitely many values of α.

Clearly the basis $\mathfrak{B}_p \subset \mathfrak{B}_b$

Therefore, the box topology is finer than the product topology.

Theorem 2.2.2.

Suppose the topology on each space X_α is given by a basis $\mathfrak{B}_\alpha$. The collection of the set of the form $\prod_{\alpha \in J} B_\alpha$, where $B_\alpha \in \mathfrak{B}_\alpha$ for each α, will save as a basis for the box topology on $\prod_{\alpha\in J} X_\alpha$. The collection of all sets of the same function form where $B_\alpha \in \mathfrak{B}_\alpha$ finitely many induces α and $B_\alpha = X_\alpha$ for all the remaining indices, will save as a basis for the product topology $\prod_{\alpha\in J} X_\alpha$.

Proof.

Let $l = \left\{\prod_{\alpha\in J} B_\alpha \in \mathfrak{B}_\alpha \text{ } is\ a\ basis\ for\ x_\alpha\right\}$ for each α.

B_α is a collection of open set in X_α for every α.

$\prod_{\alpha\in J} U_\alpha$ is open in $\prod_{\alpha\in J} X_\alpha$

Therefore l is a collection of open sets in π_{X_α}

To prove l is a basis for the box topology in $\prod_{\alpha\in J} X_\alpha$.

Now, $x = (x_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} X_\alpha$

Let U be an open set in π_{X_α} containing x

Now U is an open set in the box topology in $\pi_{X_\alpha}, x \in U$

There exists a basis element $\prod_{\alpha\in J} U_\alpha$ such that $x \in \prod_{\alpha\in J} U_\alpha \subset U$

$\Rightarrow x_\alpha \in U_\alpha$ for each α

Now, $x_\alpha \in U_\alpha$ and U_α is open in X_α and $\mathfrak{B}_\alpha$ is a basis for X_α, there exists $B_\alpha \in \mathfrak{B}_\alpha$ such that $x_\alpha \in B_\alpha \subset U_\alpha$ for each α.

Then $(x_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} B_\alpha \subset \prod_{\alpha\in J} U_\alpha \subset U$.

i.e., $x \in \prod_{\alpha\in J} B_\alpha \subset U$

For every $x \in \pi_{X_\alpha}$ and any open set U containing x there exists $\prod_{\alpha\in J} B_\alpha$ in l such that $x \in \prod_{\alpha\in J} B_\alpha \subset U$.

By Lemma 1.1.2, l is a basis for the box topology on the product space $\prod_{\alpha\in J} X_\alpha$.

Let $l' = \{\prod_{\alpha\in J} B_\alpha / B_\alpha$ for finitely many indices and $B_\alpha = X_\alpha$ for the remaining indices}

To prove l' is a basis for the product topology on $\prod_{\alpha\in J} X_\alpha$

Let $x = (x_\alpha) \in \prod_{\alpha\in J} X_\alpha$

Let V be an open set in $\prod_{\alpha \in J} X_\alpha$ containing x, there exists a basis element $\prod_{\alpha \in J} U_\alpha$ for the product topology in $\prod_{\alpha \in J} X_\alpha$ such that $x \in \prod_{\alpha \in J} U_\alpha \subset V$, where U_α is open in X_α, for $\alpha = \alpha_1, \alpha_2, \alpha_3, \dots \dots \alpha_n$ and $U_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2, \alpha_3, \dots \dots \alpha_n$

Now, $U_{\alpha i}$ is open in $X_{\alpha i}$ and $x_{\alpha i} \in U_{\alpha i}$ then there exist $B_{\alpha i} \in \mathfrak{B}_{\alpha i} \subset U_\alpha$.

Define, $\prod_{\alpha \in J} B_\alpha$ where $B_\alpha \in \mathfrak{B}_\alpha$ for $\alpha = \alpha_1, \alpha_2, \alpha_3, \dots \dots \alpha_n$

$B_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \alpha_2, \alpha_3, \dots \dots \alpha_n$

Then clearly, $\prod_{\alpha \in J} B_\alpha \in l'$ and $x = (x_\alpha)_{\alpha \in J} \in B_\alpha \subset \prod_{\alpha \in J} U_\alpha \subset V$ for all $x \in \prod_{\alpha \in J} X_\alpha$, there exists $\prod_{\alpha \in J} B_\alpha \in l'$, such that $x \in \prod_{\alpha \in J} B_\alpha \subset V$.

By Lemma 1.1.2, l' is a basis for the product topology in $\prod \quad X_\alpha$.

Example.

Consider the Euclidean space $\mathbb{R}^n$. A basis for R consists of all open intervals in $\mathbb{R}$**.** Hence, a basis for the topology of $\mathbb{R}^{\boldsymbol{n}}$ consists of all products of the form $(a_1, b_1) \times (a_2, b_2) \times \dots \dots \times (a_n, b_n)$. Since $\mathbb{R}^{\boldsymbol{n}}$ is a finite product the box and product topologies are agree. Whenever we consider $\mathbb{R}^{\boldsymbol{n}}$, we will assume that it is given this topology, unless we specifically state otherwise.

Theorem 2.2.3.

Let A_α be a subspace of X_α, for each $\alpha \in J$, then π_{A_α} is a subspace of $\prod \quad X_\alpha$ if both products are given the box topology or if both products are given the product topology.

Proof.

By theorem 2.2.1, $\prod_{\alpha \in J} B_\alpha$ is the basis for the subspace $\prod \quad A_\alpha$ ($\because A_\alpha \subset X_\alpha$)

$\therefore \prod_{\alpha \in J} A_\alpha \subset \prod_{\alpha \in J} X_\alpha$.

Theorem 2.2.4.

If each X_α is a Hausdorsff Space, that $\prod \ X_\alpha$ is a Hausdorff space in both the box and product topology.

Proof.

Since, X_α is Hausdorff, then there are distinct neighbour hoods in X_α their product also containing disjoint neighbour hood.

$\prod \ X_\alpha$ is Hausdorff.

Theorem 2.2.5.

Let $\{X_\alpha\}$ be an indexed family of space. Let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product (or) the box topology, then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

Proof.

Let $x = (x_\alpha) \in \prod \overline{A_\alpha}$

To show that $x \in \overline{\prod A_\alpha}$

Let $U = \prod U_\alpha$, be a basis for the box (or) product topology that containing x.

Since $x = (x_\alpha) \in \prod \overline{A_\alpha}$, We can choose a point $y_\alpha \subset U_\alpha \cap A_\alpha$

Then $y = (y_\alpha) \in U$ and $y \in \prod A_\alpha$

Since, U is arbitrary $(x_\alpha) \in \overline{\pi A_\alpha}$

$\therefore \pi\overline{A_\alpha} \subseteq \overline{\pi A_\alpha}$ ------------ (1)

Conversely, suppose $x = (x_\alpha) \in \overline{\pi A_\alpha}$

Such that, $x = (x_\alpha) \in \prod \overline{A_\alpha}$

Let $V = \pi_{V_\alpha} \in \pi_{X_\alpha}$ containing x

Let $V_\beta \in X_\beta$ containing x_β for each β

By the definition of product topology

Since, $\pi_\beta^{-1}(V_\beta)$ is open in $\prod_{\alpha\in J} X_\alpha$ in either topology, $x_\beta \in \ V_\beta \subset X_\beta$

Then, $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$

Since, $A_\alpha \subset X_\alpha, y_\alpha \in \prod A_\alpha$

Now, $y_\beta \in V_\beta \cap A_\beta$

Then, $x_\beta \in \overline{A_\beta}$

$\Rightarrow (x_\beta) \in \prod \overline{A_\alpha}$

$\Rightarrow \overline{\overline{\prod A_\alpha}} \subseteq \prod \overline{A_\alpha}$

$\therefore \prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

Theorem 2.2.6

Let $f: A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha: A \to X_\alpha$ for each α. Let $\prod X_\alpha$ haves the product topology. Then the function f is continuous iff each function f_α is continuous.

Proof

Let $f: A \to \prod_{\alpha \in J} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f: A \to \prod_{\alpha \in J} X_\alpha$ for each α

Let $\prod_{\alpha \in J} X_\alpha$ have the product topology

Now, let π_β be the projection of the product onto its β^{th} factor

i.e., $\pi_\beta: \prod_{\alpha \in J} X_\alpha \to X_\beta$

If U_β is open in X_β, then $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on X_α

$\therefore \pi_\beta$ is continuous

Now suppose $f: A \to \prod_{\alpha \in J} X_\alpha$ is continuous

Since, π_β and f are continuous

The composite of these two maps $\pi_\beta \circ f$ is continuous i.e., $\pi_\beta \circ f = f_\alpha$, where $f_\alpha: A \to X_\alpha$ is continuous

Conversely, suppose that each function f_α is continuous

To prove, $f: A \to \prod_{\alpha \in J} X_\alpha$ is continuous.

$\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on π_{X_α}, where U_β is open in X_β

$$f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right) = (\pi_\beta \circ f)^{-1}(U_\beta) = f_\alpha^{-1}(U_\beta)$$

Since, $f_\alpha: A \to X_\beta$ is continuous, $f_\alpha^{-1}(U_\beta)$ is open in A

$\therefore f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right)$ is open in A

$\therefore f$ is continuous.

2.3. The Metric Topology

One of the most important and frequently used ways of imposing a topology on a set is to define the topology in terms of a metric on the set. Topologies given in this way lie at the heart of modern analysis, for example. In this section, we shall define the metric topology and shall give a number of examples.

Definition.

A ***metric*** on a set *X* is a function $d: X \times X \to R$ having the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.

Given a metric d on *X*, the number $d(x, y)$ is often called the ***distance*** between *x* and *y* in the metric *d*. Given $\epsilon > 0$, consider the set $B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$ of all points *y* whose distance from *x* is less than ϵ. It is called the ***ϵ -ball centered at x***. Sometimes we omit the metric *d* from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

Definition.

If d is a metric on the set X, then the collection of all ϵ -balls $B_d(x,\epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the ***metric topology*** induced by d.

Result. Prove that the collection $\mathcal{B}$ of $\epsilon - ball$ is a basis.

Proof.

The first condition for a basis is trivial, since $x \in B(x, \epsilon)$ for any $\epsilon > 0$.

Before checking the second condition for a basis, we show that if y is a point of the basis element $B(x,\epsilon)$, then there is a basis element $B(y,\delta)$ *centered* at y that is contained in $B(x,\epsilon)$.

Now, let $y \in B(x,\epsilon)$

$\Rightarrow d(x,y) > \epsilon$

$\Rightarrow \epsilon - B(x,\epsilon) > 0$

Take $\delta = \epsilon - B(x,\epsilon)$, then $\delta > 0$.

Claim: $B(y,\delta) \subset B(x,\epsilon)$.

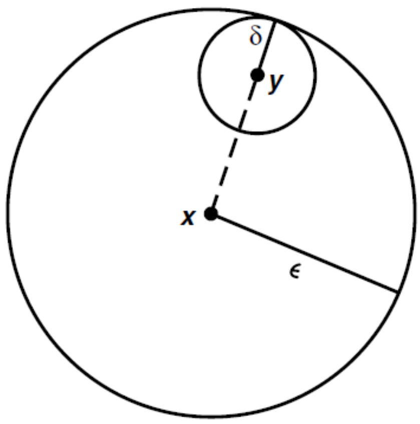


Figure 2.3.1

Let $z \in B(y,\delta)$

$\Rightarrow d(y,z) < \delta = \epsilon - d(x,y)$

$\Rightarrow d(x,y) + d(y,z) < \epsilon$

$\Rightarrow d(x,z) < \epsilon$

$\Rightarrow z \in B(x,\epsilon)$

$\therefore B(y,\delta) \subset B(x,\epsilon)$

Now to check the second condition for a basis, let B_1 and B_2 be two basis elements and let $y \in B_1 \cap B_2$. We have just shown that we can choose positive numbers δ_1 and δ_2 so that $B(y, \delta_1) \subset B_1$ and $B(y, \delta_2) \subset B_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$

Then, $B(y, \delta) \subset B_1$ and $B(y, \delta) \subset B_2$

$\Rightarrow B(y, \delta) \subset B_1 \cap B_2$.

$\therefore$ the second condition of the basis is satisfied.

Thus, the collection $\mathcal{B}$ of ϵ-ball is a basis.

Result.

A set U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

Proof.

Suppose for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

Then U is open.

Conversely, if U is open, it contains a basis element $B = B_d(x, \epsilon)$ containing *y*, and B in turn contains a basis element $B_d(y, \delta)$ centered at *y*.

Example 1. Given a set *X*, define

$$d(x, y) = 1 \ if x \neq y,$$

$$d(x, y) = 0 \ if x = y.$$

Then *d* is a metric.

For, if $x \in X$, then $B_d(x,1) = \{x\}$.

The topology it induces is the discrete topology; the basis element $B(x, 1)$ consists of the point x alone.

Example 2.

The standard metric on the real numbers $\mathbb{R}$ is defined by the equation $d(x,y) = |x - y|$. Then d is a metric and the topology it induces is the same as the order topology.

For, It is easy to check that d is a metric.

Each basis element (a, b) for the order topology is a basis element for the metric

topology; indeed, $(a,b) = B(x,\epsilon)$,where $x = (a+b)/2$ and $\epsilon = (b - a)/2$. And conversely, each ϵ -ball $B(x, \epsilon)$ equals an open interval: the interval $(x - \epsilon, x + \epsilon)$.

Definition.

If X is a topological space, X is said to be ***metrizable*** if there exists a metric d on the set X that induces the topology of X. A ***metric space*** is a metrizable space X together with a specific metric d that gives the topology of X.

Definition.

Let X be a metric space with metric d. A subset A of X is said to be ***bounded*** if there is some number M such that $d(a_1, a_2) \le M$ for every pair

a_1, a_2 of points of A. If A is bounded and nonempty, the ***diameter*** of A is defined to be the number $diam\ A = sup\{d(a_1, a_2) | a_1, a_2 \in A\}$.

Theorem 2.3.1.

Let X be a metric space with metric d. Define $\bar{d}: X \times X \rightarrow R$ by the equation

$$\bar{d}(x, y) = min\{d(x, y), 1\}.$$

Then $\bar{d}$ is a metric that induces the same topology as d.

The metric $\bar{d}$ is called the ***standard bounded metric*** corresponding to d.

Proof.

Let X be a metric space with metric d.

$\bar{d}: X \times X \rightarrow \mathbb{R}$ defined as $\bar{d}(x, y) = \min\{d(x, y), 1\}$ ... ... (1)

First two conditions for a metric are trivial.

To check the triangle inequality:

$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$ ... ... (2)

If either $d(x, y) \geq 1$ or $d(y, z) \geq 1$, then

R.H.S of (2) is atleast 1.

Since L.H.S of (2) is atmost 1, the inequality (2) holds.

Now, consider $d(x, y) < 1$ and $d(y, z) < 1$

We have $d(x, z) \leq d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z)$

Since $\bar{d}(x, z) \leq d(x, z)$ by definition, the triangle inequality holds for $\bar{d}$.

We know that the collection of $\epsilon - balls$ with $\epsilon < 1$ forms a basis for a metric topology.

Every basis element contains x such that an ϵ-ball centered at x.

Since the collection of ϵ-balls with $\epsilon < 1$under d and $\bar{d}$ are same.

Thus d and $\bar{d}$ induce the same topology on X.

Definition.

Given $\boldsymbol{x} = (x_1, \ldots, x_n)$ in R^n, we define the ***norm*** of **x** by the equation

$$||\boldsymbol{x}|| = (x_1^2 + \cdots + x_n^2)^{1/2}$$

and we define the **E*uclidean metric*** d on R^n by the equation

$$d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}|| = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{1/2}.$$

We define the ***square metric*** ρ by the equation

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

Remark.

(i) d is a metric on $\mathbb{R}^n$.

(ii) ρ is a metric on $\mathbb{R}^n$.

Proof.

(i)Since each$(x_i - y_i)^2, i = 1,2, \ldots, n$ is positive

We have $d(x, y) = ||x - y||$

$$= \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} \geq 0 \ \forall x, y \in \mathbb{R}^n$$

Also $d(x,y) = 0\ iff\ ||x-y||$=0

$$iff\ \left[\sum_{i=1}^{n}(x_i-y_i)^2\right]^{\frac{1}{2}} = 0$$

$$iff\ (x_i-y_i)^2, i = 1,2,\dots,n$$

$$iff\ (x_i-y_i) = 0,, i = 1,2,\dots,n$$

$$iff\ x_i = y_i, i = 1,2,\dots,n$$

$$iff\ x = y$$

And

$$d(x,y) = ||x-y||$$

$$= \left[\sum_{i=1}^{n}(x_i-y_i)^2\right]^{1/2}$$

$$= \left[\sum_{i=1}^{n}(y_i-x_i)^2\right]^{1/2}$$

$$= ||y-x||$$

$$= d(y,x)$$

Now, $d(y,x) = ||x-z||$

$$= ||x-y+y-z||$$

$$\leq\ ||x-y|| + ||y-z||$$

$$= d(x,y) += d(y,z)$$

Thus, d is a metric on $\mathbb{R}^n$.

(ii)since $|x_i - y_i| \geq 0\ \forall\ i = 1,2,\dots n$

We have $\max\{|x_1 - y_1|, \dots . |x_n - y_n| \} \geq 0$

$$\Rightarrow \rho(x, y) \geq 0$$

Also, $\rho(x, y) = 0 \; iff \; \max\{|x_1 - y_1|, \dots . |x_n - y_n| \} = 0$

$$iff \; |x_i - y_i| = 0, i = 1,2, \dots , n$$

$$iff \; x_i = y_i \, , i = 1,2, \dots , n$$

$$iff \; x = y$$

And, $\rho(x, y) = \max\{|x_1 - y_1|, \dots . |x_n - y_n| \}$

$$= \max\{|y_1 - x_1|, \dots . |y_n - x_n| \}$$

$$= \rho(y, x)$$

For each $i = 1,2, \dots , n$, we have

$$|x_i - z_i| = |x_i - y_i + y_i - z_i|$$

$$\leq |x_i - y_i| + |y_i - z_i|$$

$$max|x_i - z_i| \leq \max\{|x_i - y_i| + |y_i - z_i|\}$$

We have, $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

Lemma 2.3.2.

Let d and $\bar{d}$ be two metrics on the set *X*; let $\mathcal{T}$ and $\mathcal{T}'\_$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each *x* in *X* and each $\epsilon > 0$, there exists a $\delta > 0$ such that $B_d(x, \delta) \subset B_d(x, \epsilon)$.

Proof.

Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$.

Given the basis element $B_d\ (x, \epsilon\ )$ for $\mathcal{T}$, there is, by Lemma 1.2.3, a basis element $\mathcal{B}'$ for the topology $\mathcal{T}'$ such that $x \in \mathcal{B}'\ \subset B_d(x, \epsilon)$.

Within $\mathcal{B}'$ we can find a ball $B_d(x, \delta)$ centered at x.

Hence $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Conversely, suppose the $\delta - \epsilon$ condition holds.

$i.e., B_{d'}(x, \delta) \subset B_d(x, \epsilon)\ \ \dots\dots(*)$.

Given a basis element B for $\mathcal{T}$ containing x, we can find within B a ball $B_d(x, \epsilon)$ centered at x.

By the given condition (*), there is a δ such that $B_{d\prime}(x, \delta) \subset B_d(x, \epsilon)$.

Then Lemma 1.2.3, applies to show $\mathcal{T}'$ is finer than $\mathcal{T}$.

Theorem 2.3.3.

The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$ (or) $\mathbb{R}^n$ is metrizable.

Proof.

Step 1:

Let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ be two points of $\mathbb{R}^n$.

First we prove that $\rho(\boldsymbol{x}, \boldsymbol{y})\ \leq\ d(\boldsymbol{x}, \boldsymbol{y})\ \leq \sqrt{n}\rho(\boldsymbol{x}, \boldsymbol{y})$.

We know that $d(x, y) = \|x - y\| = [\sum_{i=1}^{n}(x_i - y_i)^2]^{1/2}$

And $\rho(x, y) = \max\{|x_1 - y_1|, \dots . |x_n - y_n|\}$

Always $|x_i - y_i| \leq [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{1}{2}}$

$\Rightarrow \max\{|x_1 - y_1|, \ldots . |x_n - y_n| \} \leq d(x, y)$

$\Rightarrow \rho(\boldsymbol{x}, \boldsymbol{y}) \leq d(x, y)$.

Now, $(x_i - y_i)^2 \leq \left(\rho(x, y)\right)^2, \forall i = 1,2, \ldots, n$

Adding the above inequality we get,

$$(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2$$
$$\leq \left(\rho(x, y)\right)^2 + \left(\rho(x, y)\right)^2 + \cdots + \left(\rho(x, y)\right)^2$$
$$\leq n\left(\rho(x, y)\right)^2$$

Square root on both sides we get,

$$\left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}} \leq \sqrt{n} \left(\rho(x, y)\right)^2$$

Hence, $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$.

Step 2: To prove the two metric topologies are the same.

Let $\mathcal{T}_d$ and $\mathcal{T}_\rho$ be the topologies induced by d and ρ respectively.

Prove that $\mathcal{T}_d \supset \mathcal{T}_\rho$.

Let $x \in X$ and $\epsilon > 0$ be given.

Consider $B_\rho(x, \epsilon)$ and take $\delta = \epsilon$.

Claim: $B_d(x, \epsilon) \subset B_\rho(x, \epsilon)$

Let $y \in B_d(x, \epsilon)$

$\Rightarrow d(x, y) < \epsilon$

We have $\rho(x, y) \leq d(x, y)$ (by step 1)

$\therefore\ \rho(x,y) < \epsilon$

$\Rightarrow y \in B_\rho(x,\epsilon)$

Thus $B_d(x,\epsilon) \subset B_\rho(x,\epsilon)$.

By Lemma 2.3.1, $\mathcal{T}_d \supset \mathcal{T}_\rho \ldots.(1)$

Claim: $\mathcal{T}_\rho \supset \mathcal{T}_d$

Let x ∈ X and $\epsilon > 0$ be given

Consider $B_d(x,\epsilon)$ and take $\delta = \frac{\epsilon}{\sqrt{n}}$.

Claim: $B_\rho(x,\delta) \subset B_d(x,\epsilon)$.

Let $y \in B_\rho(x,\delta)$

$\Rightarrow \rho(x,y) < \delta = \frac{\epsilon}{\sqrt{n}}$

$\Rightarrow \sqrt{n}\rho(x,y) \leq \epsilon$

We have $d(x,y) \leq \sqrt{n}\,\rho(x,y)$ (step 1)

$\Rightarrow d(x,y) < \epsilon$

$\Rightarrow y \in B_d(x,\epsilon)$

$\therefore B_\rho(x,\epsilon) \subset B_d(x,\delta)$

By Lemma 2.3.1, $\mathcal{T}_\rho \supset \mathcal{T}_d \ldots..(2)$

From (1) and (2), $\mathcal{T}_\rho = \mathcal{T}_d$

The metric topologies induced by d and ρ are the same.

Step 3: Prove that the product topologies on $\mathbb{R}^n$ is the same as the metric topology induced by ρ.

Let $B = (a_1, b_1) \times \cdot \cdot \cdot \times (a_n, b_n)$ be a basis element for the product topology, and let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an element of B.

For each i, there is an ϵ_i such that $(x_i - \epsilon_i , x_i + \epsilon_i) \subset (a_i , b_i)$.

choose $\epsilon = min\{\epsilon_1, \ldots, \epsilon_n\}$.

Then $B_\rho(\boldsymbol{x}, \epsilon) \subset B$

By Lemma 2.3.1, $\mathcal{T}_\rho \supset \mathcal{T}$

Conversely, let $B_\rho(\boldsymbol{x}, \epsilon)$ be a basis element for the ρ-topology.

Given the element $\boldsymbol{y} \in B_\rho(\boldsymbol{x}, \epsilon)$, we need to find a basis element B for the product topology such that $\boldsymbol{y} \in B \subset B_\rho(\boldsymbol{x}, \epsilon)$.

Now, $B_\rho(\boldsymbol{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \ldots \times (x_n - \epsilon, x_n + \epsilon)$ is itself a basis element for the product topology.

$$\therefore \mathcal{T} \supset \mathcal{T}_\rho$$

Hence $\mathcal{T} = \mathcal{T}_\rho$

Thus, the product topology on $\mathbb{R}^n$ is the same as the metric topology by ρ.

Definition.

Given an index set J , and given points $\boldsymbol{x} = (x_\alpha)_{\alpha \in J}$ and $\boldsymbol{y} = (y_\alpha)_{\alpha \in J}$ of R^J , let us define a metric $\bar{\rho}$ on R^J by the equation

$$\bar{\rho}(\boldsymbol{x}, \boldsymbol{y}) = sup\{\bar{d}(x_\alpha, y_\alpha) | \alpha \in J\},$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$.Then ρ is called the ***uniform metric*** on R^J , and the topology it induces is called the ***uniform topology***.

The relation between this topology and the product and box topologies is the following:

Theorem 2.3.4.

The uniform topology on R^J is finer than the product topology and coarser than the box topology; these three topologies are all different if J is infinite.

Proof.

Suppose that we are given a point **x** = $(x\ \alpha)\ \alpha \in J$ and a product topology basis

Element $\prod U_\alpha$ about **x**.

Let $\alpha_1, \ldots, \alpha_n$ be the indices for which $U_\alpha = \mathbb{R}$.

Since U_{α_i} is open in $\mathbb{R}$, for each i choose $\epsilon_i > 0$ such that $B_{\bar{d}}(x_{\alpha_i}, \epsilon_i) \subset U_{\alpha_i}$.

Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$

$\Rightarrow B_{\bar{\rho}}(x, \epsilon) \subset \prod U_\alpha$

If $z \in \mathbb{R}^J$ such that $\bar{\rho}(x, z) < \epsilon$

$\Rightarrow \bar{d}(x_\alpha, z_\alpha) < \epsilon\ \forall \alpha$

Hence uniform topology is finer than the product topology.

On the other hand, let B(x, ϵ) in the $\bar{\rho} - metric$.

Then the box neighbourhood $U = \prod \left(x_\alpha - \frac{\epsilon}{2}, x_\alpha + \frac{\epsilon}{2}\right)$ of **x** is contained in B.

For if $\boldsymbol{y} \in U$, then $\bar{d}(x_\alpha, y_\alpha) < \frac{\epsilon}{2}$ for all α, so that $\bar{\rho}(\boldsymbol{x}, \boldsymbol{y}) \leq \frac{\epsilon}{2}$.

Suppose J is infinite.

Let $x = (x_\alpha)_{\alpha \in J} \in \mathbb{R}^J$ and let $\epsilon > 0$ be given.

$B_{\bar{\rho}}(x, \epsilon) = \{y \mid \bar{\rho}(x, y) < \epsilon\}$

$= \{y \mid \bar{d}(x_\alpha, y_\alpha) < \epsilon_\alpha \ \forall \alpha\}$

$= \{y \mid |x_\alpha - y_\alpha| < \epsilon \quad \forall \alpha\}$

$= \{y \mid |y_\alpha - x_\alpha| < \epsilon \quad \forall \alpha\}$

$= \{y \mid -\epsilon < y_\alpha - x_\alpha < \epsilon \ \forall \alpha\}$

$= \{y \mid x_\alpha - \epsilon < y_\alpha < x_\alpha + \epsilon \ \forall \alpha\}$

$= (x_{\alpha_1} - \epsilon, x_{\alpha_1} + \epsilon) \times (x_{\alpha_2} - \epsilon, x_{\alpha_2} + \epsilon) \times \ldots . \times (x_{\alpha_n} - \epsilon, x_{\alpha_n} + \epsilon)$

This is a basis element for the uniform topology but we cannot find a basis element $\prod U_\alpha$.

For the product such that $\prod X_\alpha \subset B_{\bar{\rho}}(x, \epsilon)$

$\therefore$ the product topology is not finer than the uniform topology (since in $\prod U_\alpha$, U_α is open in R for only finite number of indices α).

$\therefore$ they are different.

Theorem 2.3.5.

Let $\bar{d}(a, b) = min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^\omega$, define $D(x, y) = lub\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$. Then D is a metric that induces the product topology on $\mathbb{R}^\omega$ is metrizable.

Proof

Step 1:

First, we prove that D is a metric on $\mathbb{R}^\omega$

i) Let $x, y \in \mathbb{R}^\omega$

Then $\bar{d}(x_i, y_i) \geq 0$

$\Rightarrow \frac{\bar{d}(x_i,y_i)}{i} \geq 0 \forall i$

$\Rightarrow lub\left(\frac{\bar{d}(x_i,y_i)}{i}\right) \geq 0\ \forall i$

$\Rightarrow D(x, y) \geq 0$

Also $D(x, y) = 0$

$\Leftrightarrow lub\left(\frac{\bar{d}(x_i,y_i)}{i}\right) = 0$

$\Leftrightarrow \frac{\bar{d}(x_i,y_i)}{i} = 0$

$\Leftrightarrow \bar{d}(x_i, y_i) = 0\ \forall i$

$\Leftrightarrow |x_i - y_i| = 0\ \forall i$

$\Leftrightarrow x_i = y_i\ \ \forall i$

$\Leftrightarrow x = y$

(ii) $D(x, y) = lub\left(\frac{\bar{d}(x_i,y_i)}{i}\right)$

$= lub\ \left(\frac{\bar{d}(x_i,y_i)}{i}\right)$

$= D(y, x)$

(iii) Let $x, y, z \in \mathbb{R}^\omega$

For each i,

$$\bar{d}(x_i, z_i) \leq \bar{d}(x_i, y_i) + \bar{d}(y_i, z_i)$$

$$\Rightarrow \frac{\bar{d}(x_i,z_i)}{i} \leq \frac{\bar{d}(x_i,y_i)}{i} + \frac{\bar{d}(y_i,z_i)}{i}$$

Always, $\frac{\bar{d}(x_i,y_i)}{i} \leq D(x, y) + D(y, z)$

Hence D is a metric on $\mathbb{R}^w$

Step 2:

Let $\mathcal{T}$ be the product topology on $\mathbb{R}^\omega$ and Let $\mathcal{T}_D$ be the metric topology induced by D

Prove that, $\mathcal{T} = \mathcal{T}_D$

First, we prove that $\mathcal{T} \supset \mathcal{T}_D$

Let U be open in the metric topology $\mathcal{T}_D$ and

Let $x \in U$.

To prove, $\mathcal{T} \supset \mathcal{T}_D$ it is enough to find an open set V in the product topology such that $x \in V \subset U$.

Since U is open in the metric topology and $x \in U$ we can choose an $\varepsilon -$ ball $B_D(x, \varepsilon)$ such that $B_D(x, \varepsilon) \subset U$

Then choose N large enough such that $\frac{1}{N} < \varepsilon$

Let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times \dots \dots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \dots.$

Then V is a basis element for the product topology

Prove that, $V \subseteq B_D(x, \varepsilon)$

Let y be any point of $\mathbb{R}^\omega$

Now, $i \geq N \Rightarrow \frac{1}{i} < \frac{1}{N}$

By definition, $\bar{d}(x_i, y_i) \leq 1 \ \forall i$

$$\Rightarrow \frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{i} \ \forall i$$

$$\Rightarrow \frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N} \ if i \geq N$$

$$D(x, y) \leq \max\left\{\bar{d}\left(\frac{x_1, y_1}{1}\right), \bar{d}\left(\frac{x_2, y_2}{2}\right), \dots \dots \bar{d}\left(\frac{x_N, y_N}{N}\right), \frac{1}{N}\right\}$$

Claim: $\quad V \subset B_D(x, \varepsilon)$

Let $y \in V$

Then $y_i \in (x_i - \varepsilon, x_i + \varepsilon)$ for $i = 1,2 \dots \dots N$

$\Rightarrow x_i - \varepsilon < y_i < x_i + \varepsilon$

$\Rightarrow |x_i - y_i| < \varepsilon \; \forall i = 1,2, \dots . N$

$\Rightarrow \bar{d}(x_i, y_i) \leq |x_i - y_i| < \varepsilon \; \forall i = 1,2 \dots \dots N$

$\Rightarrow \bar{d}(x_i, y_i) < \varepsilon \leq i\varepsilon \; for \; i = 1,2, \dots \dots N$

Now $\Rightarrow \frac{\bar{d}(x_i,y_i)}{i} < \varepsilon$ for $i = 1,2, \dots . . N$

$$\frac{\bar{d}(x_i,y_i)}{i} \leq \frac{1}{N} \quad i \geq N \quad \rightarrow (1)$$

$\Rightarrow \frac{\bar{d}(x_i,y_i)}{i} < \varepsilon$ for $i \geq N \; \rightarrow (2) \quad \left[\because \frac{1}{N} < \varepsilon\right]$

From (1) and (2),

$$\frac{\bar{d}(x_i,y_i)}{i} < \varepsilon \; \forall i$$

$\Rightarrow lub \left\{\frac{\bar{d}(x_i,y_i)}{i}\right\} < \varepsilon$

$\Rightarrow D(x, y) < \varepsilon$

$\Rightarrow y \in B_D(x, \varepsilon)$

Hence $V \subset B_D(x, \varepsilon) \subset U$

$\therefore V \subset U$

Hence $\mathcal{T} \supset \mathcal{T}_D$

Step 3:

We have to prove that $\mathcal{T}_D \supset \mathcal{T}$

Let $x \in \mathbb{R}^\omega$

Consider the basis element $U = \prod_{i \in \mathbb{Z}} U_i$ containing x for the product topology where U_i is open in $\mathbb{R}$ for $i = d_1, d_2, \dots \dots d_n$ and $U_i = \mathbb{R}$ for all other values of 1.

To prove, $\mathcal{T}_D \supset \mathcal{T}$

It is enough to prove that an open set V for the metric topology such that, $x \in V \subset U$

Since, V_i is open is $\mathbb{R}$ and $x_i \in V_i$

We can choose ε_i such that $(x_i - \varepsilon, x_i + \varepsilon) \subset U_i$ for $i = d_1, d_2, \dots \dots d_n$

Also choose each $\varepsilon_i \leq 1$ then define

$$\varepsilon = \min\left\{\frac{\varepsilon_i}{i} / i = d_1, d_2, \dots \dots d_n\right\}$$

Claim: $B_D(x, \varepsilon) \subset U$

Let $y = \in B_D(x, \varepsilon)$

$$D(x, y) < \varepsilon$$

But $\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \in \ \forall i$

$$\Rightarrow \frac{\bar{d}(x_i, y_i)}{i} < \varepsilon \ \forall i$$

If $i = d_1, d_2, \dots \dots d_n$, then $\varepsilon \leq \frac{\varepsilon_i}{i}$

Hence for $i = d_1, d_2, \dots \dots d_n$ we have

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{\varepsilon_i}{i}$$

$$\Rightarrow \bar{d}(x_i, y_i) < \varepsilon_i$$

But $\varepsilon_i \leq 1$ we have

$$\bar{d}(x_i, y_i) = |x_i - y_i|$$

$$\therefore |x_i - y_i| < \varepsilon_i$$

$\Rightarrow y_i \in (x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i$ for each $i = d_1, d_2, \dots \dots d_n$

$\Rightarrow y_i \in U_i$ for each $i = d_1, d_2, \dots \dots d_n$

$$\Rightarrow y \subset \pi U_i$$

$$\Rightarrow y \in U$$

$$\therefore \mathcal{T}_D \supset \mathcal{T}$$

Thus $\mathcal{T}_D \supset \mathcal{T}$

$\therefore D$ is a metric that indues that the product topology of $\mathbb{R}^\omega$

$\therefore \mathbb{R}^{\omega}$ is metrizable.

2.4. The Metric Topology (Continued)

Theorem 2.4.1.

Let $f: X \to Y$. Let x and y be metrizable with metrices d_X and d_Y respectively. The continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x, y) < \delta \quad \Rightarrow d_Y\big(f(x), f(y)\big) < \varepsilon$.

Proof.

Suppose that $f: X \to Y$ is continuous.

To Prove $d_X(x, y) < \delta \Rightarrow d_Y\big(f(x), f(y)\big) < \epsilon$.

Given x & ε, Consider the $f^{-1}(B(f(x), \varepsilon))$, which is open in X and contains the point x.

It contains some δ -ball B(x, δ) centered at x.

If y is in this δ-ball, then *f (y)* is in the ε -ball centered at *f (x)*

$\therefore\ d_X(x, y) < \delta \Rightarrow d_Y\big(f(x), f(y)\big) < \varepsilon$

Conversely, suppose that the $\varepsilon - \delta$ condition is satisfied.

To prove f is continuous.

Let V be an open set in Y

Claim: $f^{-1}(V)$ is open in X

Let $x \in f^{-1}(V)$

Since $f(x) \in V$,

Since V is open, there is an ε-ball $B(f(x), \varepsilon)$ centered at $f(x)$ and contained in V

By $\varepsilon - \delta$ condition, there exists $\delta - ball\ B(x, \delta)$such that $f\big(B_X(x, \delta)\big) \subseteq B_{d_Y}(f(x), \varepsilon)$

$\therefore f\left(B_{d_y}(x, \delta)\right) \subset V$

$\Rightarrow B_{d_x}(x,\delta) \subset f^{-1}(V)$

$\Rightarrow f^{-1}(V)$ is open in V

$\therefore f$ is continuous.

Note.

The $\varepsilon - \delta$ condition equivalent to $y \in B_{d_X}(x,\delta) \Rightarrow f(y) \in B_{d_Y}(f(x),\varepsilon)$. Also, the condition is equivalent to $f\left(B_{d_X}(x,\delta)\right) \leq B_{d_Y}(f(x),\varepsilon)$

Note. A sequence of points of X is a function mapping from $\mathbb{Z}_+$ onto X

Theorem 2.4.2 (The sequence Lemma)

Let X be a topological space. Let $A \subset X$. If there is a sequence of points of A converging to x. Then $x \in \bar{A}$; the converse holds if X is metrizable.

Proof

Suppose that $(x_n) \to x$, where $x_n \in A$

To prove $x \in \bar{A}$

Let U be a neighbour hood of x

Since $(x_n) \to x$, there exist a positive integer N such that $x_i \in U \quad \forall i \geq N$

Since, $x_i \in A \quad \forall i$ we have $\quad x_i \in U \cap A \quad \forall i \geq N$

$\Rightarrow U$ intersects A

$\Rightarrow x \in \bar{A}$

Conversely, suppose that X is mertizable and $x \in \bar{A}$

Let d be a metric for a topology of X.

For each positive integer *n*, take the neighborhood $B_d\ (x, 1/n)$ of radius 1/*n* of *x*, and choose $x_n \in B\left(x, \frac{1}{n}\right) \cap A$.

Claim: $(x_n) \to x$

Any open set *U* containing *x* contains an ε -ball *Bd (x, _)* centered at *x*;

choose *N* so that $1/N < \varepsilon$, then *U* contains x_i for all $i\ \geq\ N$

$$\therefore n \geq N\ \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

$$\Rightarrow B_d\left(x, \frac{1}{n}\right) \subset B_d(x, \varepsilon) \subset U$$

But, $x_n \in B_d\left(x, \frac{1}{n}\right)\ \ \forall n \geq N$

Hence, $n \geq N,\ \ x_n \in U$

$$\therefore (x_n) \to x$$

Hence the theorem.

Theorem 2.4.3.

Let $f: X \to Y$. Let X be metrizable the function f is continuous then for every convergent sequence $(x_n) \to x$ in X, the sequence $f(x_n) \to f(x)$. The converse holds if *X* is metrizable.

Proof.

Suppose that f is continuous

Given $(x_n) \to x$

To prove $f(x_n) \to f(x)$

Let V be a neighbour hood of $f(x)$

Since f is continuous, $f^{-1}(V)$ is a neighbourhood of x

Since, $(x_n) \to x$, there exist a N such that $x_n \in f^{-1}(V)\ \ \forall n \geq N$

Then $f(x_n) \in V\quad \forall n \geq N$

$\therefore f(x_n) \to f(x)$

Conversely, suppose that for every convergent sequence $x_n \to x$ in X, $f(x_n) \to f(x)$

To prove: f is continuous

Let A be a subset of X

To prove f is continuous, it is enough to prove that $f(\bar{A}) \subseteq \overline{f(A)}$

Let $x \in \bar{A}$

Then $f(x) \in f(\bar{A})$

Claim: $f(x) \in \overline{f(A)}$

By the sequence Lemma, there is a sequence (x_n) of points of A such that, $x_n \to x$

By hypothesis $f(x_n) \to f(x)$

i.e., $\big(f(x_n)\big)$ is a sequence of points of $f(A)$ such that $f(x_n) \to f(x)$

By the sequence lemma, $f(x) \in \overline{f(A)}$

$$\Rightarrow f(\bar{A}) \subset \overline{f(A)}$$

Hence f is continuous.

Lemma 2.4.4.

The addition, subtraction and multiplication operations are continuous function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ and the quotient operation is a continuous function form $\mathbb{R} \times \mathbb{R} - \{0\}$ into $\mathbb{R}$.

Proof

We know that the function $f: X \to Y$, where X and Y are metrizable with metric d_X and d_Y respectively is continuous iff given $x \in X$ and given $\varepsilon > 0$, there exist $\delta > 0$ such that $d_x(x, y) < \delta \Rightarrow d_y\big(f(x), f(y)\big) < \varepsilon$ and also consider the metric $d(a, b) = |a - b|$ on $\mathbb{R}$ and the metric on $\mathbb{R}^2$ is given by

$$f\big((x, y), (x_0, y_0)\big) = \max\{|x - x_0|, |y - y_0|\}$$

i) Show that the addition '+' is continuous

Here '+' is a function from $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

Let $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and Let $\varepsilon > 0$ be given

Take $\delta = \frac{\varepsilon}{2}$

Then $\delta > 0$

Now $d(x + y, x_0 + y_0) = |(x + y) - (x_0 + y_0)|$

$\leq |x - x_0| + |y - y_0|$

And

$\rho\big((x, y), (x_0, y_0)\big) < \delta$

$\Rightarrow |x - x_0| < \delta$ and $|y - y_0| < \delta$

$\therefore d(x + y, x_0 + y_0) < \delta + \delta = 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$

$\therefore \rho\big((x, y), (x_0, y_0)\big) < \delta \Rightarrow d(x + y, x_0 + y_0) < \varepsilon$

Thus '+' is continuous.

ii) Show that the subtraction '-' is continuous

Here '-' is a function from $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

Let $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and let $\varepsilon > 0$ be given

Take $\delta = \frac{\varepsilon}{2}$

Then $\delta > 0$

Now, $d(x - y, x_0 - y_0) = |(x - y) - (x_0 - y_0)|$

$= |(x - x_0) + (y_0 - y)|$

$= |(x - x_0)| + |(y - y_0)|$

And

$\rho\big((x, y), (x_0, y_0)\big) < \delta$

$\Rightarrow |x - x_0| < \delta$ and $|y - y_0| < \delta$

$\therefore d(x-y, x_0 - y_0) < \delta + \delta = 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$

$\therefore \rho\big((x,y),(x_0,y_0)\big) < \delta \Rightarrow d(x-y, x_0-y_0) < \varepsilon$

Thus '-' is continuous.

iii) show that multiplication is continuous

let $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and let $\varepsilon > 0$ be given

Take $3\delta = \min\{\frac{\epsilon}{|x_0|+|y_0|+1}, \sqrt{\epsilon}\}$

$\Rightarrow 3\delta < \frac{\epsilon}{|x_0|+|y_0|+1}$ $and\ 3\delta < \sqrt{\epsilon}$

$\Rightarrow \delta < \frac{1}{|x_0|+|y_0|+1}\left(\frac{\epsilon}{3}\right)$ and $< \frac{\sqrt{\epsilon}}{3}, \delta^2 < \frac{\epsilon}{9}$

Now, $\rho\big((x,y),(x_0,y_0)\big) < \ \delta$

$\Rightarrow |x - x_0| < \delta$ and $|y - y_0| < \varepsilon$

$\therefore d(xy, x_0y_0)) = |xy - x_0y_0|$

$= |xy - x_0y_0 + xy_0 - xy_0 + x_0y - x_0y + x_0y_0 - x_0y_0|$

$= |x_0(y - y_0) + y_0(x - x_0) + (x - x_0)(y - y_0)|$

$\leq |x_0||y - y_0| + |y_0||x - x_0| + |x - x_0||y - y_0|$

$\leq |x_0|\delta + |y_0|\delta + \delta^2$

$= (|x_0| + |y_0|)\delta + \delta^2$

$< \frac{(|x_0|+|y_0|)}{|x_0|+|y_0|+1}\left(\frac{\varepsilon}{3}\right) + \frac{\varepsilon}{9}$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{9}$

$= \frac{4\varepsilon}{9} < \varepsilon$

i.e., $d(xy, x_0y_0) < \varepsilon$

Thus, $\rho\big((x,y),(x_0,y_0)\big) < \delta \Rightarrow d(xy, x_0y_0) < \varepsilon$

$\therefore$ Multiplication is continuous.

iv) show that the operation taking reciprocals is continuous map for $\mathbb{R}\{\mathbf{0}\}$ **to** $\mathbb{R}$

let $x_0 \in \mathbb{R}\{0\}$ and let $\varepsilon > 0$ be given

Then $x_0 \neq 0$

Take, $\delta = \min\{\frac{|x_0|}{2}, \frac{x_0^2\varepsilon}{2}\}$

Now, $d\left(\frac{1}{x}, \frac{1}{x_0}\right) = \left|\frac{1}{x} - \frac{1}{x_0}\right|$

$= \left|\frac{x_0 - x}{xx_0}\right|$

$\frac{|x-x_0|}{|xx_0|}$

If $|x - x_0| < \delta$

Then, $|xx_0 - x_0^2| = |x_0(x - x_0)|$

$= |x_0||x - x_0|$

$< |x_0|\delta$

$< |x_0|\frac{|x_0|}{2}$

$= \frac{|x_0|^2}{2}$

$= \frac{x_0^2}{2}$

$\Rightarrow \frac{-x_0^2}{2} < xx_0 - x_0^2 < \frac{x_0^2}{2}$

$x_0^2 - \frac{x_0^2}{2} < xx_0 < x_0^2 + \frac{x_0^2}{2}$

$\frac{x_0^2}{2} < xx_0 < \frac{3x_0^2}{2}$

$\therefore 0 < \frac{x_0^2}{2} < xx_0$

$$\Rightarrow \frac{1}{xx_0} < \frac{2}{x_0^2}$$

Hence, $d\left(\frac{1}{x}, \frac{1}{x_0}\right) = \frac{|x-x_0|}{|xx_0|}$

$$< \frac{\delta}{|xx_0|}$$

$$< \frac{2\delta}{x_0^2}$$

$$< \varepsilon$$

$$\therefore d(x, x_0) < \delta \Rightarrow d\left(\frac{1}{x}, \frac{1}{x_0}\right) < \varepsilon$$

Hence the reciprocal operation in continuous.

v) show that the quotation is continuous

Now, $\frac{x}{y} = (x)\left(\frac{1}{y}\right)$

Since, the multiplication and the reciprocal operation are continuous, the quotient operation is continuous.

Theorem 2.4.5.

If X is a topological space, and if $f, g: X \to \mathbb{R}$ are continuous. Then $f + g, f - g$ and fg are continuous. If $g(x) \neq 0, \forall x$, then $\frac{f}{g}$ is continuous.

Proof.

Let X be a topological space and $f, g: X \to \mathbb{R}$ are continuous.

Define $h: X \to \mathbb{R} \times \mathbb{R}$ by $h(x) = \big(f(x), g(x)\big)$

Since, f and g are continuous, then h is also continuous

Now, $f + g = h \circ f$

$\Rightarrow f + g$ is composition of two continuous function

$\Rightarrow f + g$ is continuous

Similarly, $f - g = - \circ h, f.g = . \circ h, \frac{f}{g} = \div \circ h$ are continuous.

Definition.

Let $f_n: X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y, we say that the sequence $\{f_n\}$ ***converges uniformly*** to the function $f: X \to Y$ if given $\varepsilon > 0$, there exist an integer N, such that $d\big(f_n(x), f(x)\big) < \varepsilon, \ \forall n \geq N$ and $\forall x \in X$

Theorem 2.4.6 (Uniform limit theorem)

Let function $f_n: X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

Proof.

Let V be an open in Y

Claim: $f^{-1}(V)$ is open in X

Let $x_0 \in f^{-1}(V)$

$\Rightarrow f(x_0) \in V$

To prove f is continuous

It is enough to find the neighbour hood U of x_0 such that $f(U) \subset V$

Let $y_0 = f(x_0)$, then $y_0 \in V$

Since V is open in Y, we can choose an $\varepsilon -$ball $B_d(y_0, \varepsilon)$ such that $B_d(y_0, \varepsilon) \subset V$

i.e., $B_d(f(x_0), \varepsilon) \subset V \quad \ldots.(1)$

Since, $\{f_n\}$ is converges uniformly to f we can choose N such that $d(f_n(x), f(x) < \frac{\varepsilon}{3} \ \forall n \geq N, \forall x \in X \ldots\ldots.(2)$

Consider, $B_d\left(f_n(x_0), \frac{\varepsilon}{3}\right)$

Since f_N is continuous we can choose a neighbourhood U of x_0 such that $f_n(U) \subset B_d\left(f_N(x_0), \frac{\varepsilon}{3}\right)$ …… (3)

Claim: $f(U) \subset B_d(f_N(x_0), \varepsilon)$

Let $x \in U \Rightarrow f(x) \in f(V)$

Now, $d(f(x), f(x_0)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_0)) + d(f_n(x_0), f(x_0))$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{3\varepsilon}{3} = \varepsilon$$

i.e., $d(f(x), f(x_0)) < \in$

$\Rightarrow f(x) \in B_d(f(x_0), \varepsilon) < \varepsilon$

$\Rightarrow f(x) \in V$

$\Rightarrow f(U) \subset V$

Hence f is continuous.

Example 1

Show that $\mathbb{R}^{\omega S}$ in the box topology is not metrizable.

Solution

We prove that the sequence lemma does not hold for $\mathbb{R}^{\omega}$

Let $A = \{(x_1, x_2, \ldots.)/x_i > 0, \forall i\}$ be a subset of $\mathbb{R}^{\omega}$

To prove $O \in \bar{A}$ where $O = (0,0, \ldots..)$

Let $B = (a_1, b_1) \times (a_2, b_2) \times \ldots\ldots$ be any basis element containing zero

$\Rightarrow B$ intersects A that implies $O \in \bar{A}$

Now, we prove that there is no sequence of points of A converging to O

Let $\{a_n\}$ be a sequence of point of A where $a_n = \{x_{1n,} x_{2n}, \ldots..)$ here each $x_{in} \geq 0$

Let $B' = (-x_{11}, x_{11}) \times (-x_{22} \times x_{22}) \times \ldots\ldots$

Since each $x_{ii} > 0, o \in B'$

Hence B' is also a basis element for the box topology containing zero.

Claim: (a_n) doesnot belong to B' $\forall n$

Then n[th] coordinate of $a_n = x_{nn} \notin (-x_{nn}, x_{nn})$

$\Rightarrow x_{nn} \notin B' \forall n$

Hence the $\{a_n\}$ cannot converges to zero in the box topology.

$\therefore$ By sequence lemma, $\mathbb{R}^\omega$ is not metrizable in the box topology.

Example 2.

Show that an uncountable product of $\mathbb{R}$ with itself is not metrizable.

Solution.

Let J be an uncountable index set.

Prove that, $\mathbb{R}^J$ doesnot satisfies the sequence lemma in the product topology

Let $A = \{(x_\alpha)/x_\alpha = 0$, for infinitely many value of α and $x_\alpha = 1, \forall$ other value of $x\}$ be a subset of $\mathbb{R}^J$

Claim: $\bar{0} \in \bar{A}$ where $\bar{0} = (0,0, \dots \dots)$

Let $\prod U_\alpha$ be a basis element containing $\bar{0}$

Then, $U_\alpha \neq R$ for finitely many values of d. Say $d = d_1, d_2 \dots \dots d_n$

Construct a point (x_α) such that

$x_\alpha = 0$ if $d = d_1, d_2 \dots \dots d_n$ and

$x_\alpha = 1$ if $d \neq d_1, d_2 \dots \dots d_n$

Since, $0 \in \pi_{U_\alpha}, 0 \in U_\alpha$ for $d = d_1, d_2 \dots \dots d_n$

$\Rightarrow (x_\alpha) \in \pi_{U_\alpha}$

By construct of $A, (x_\alpha) \in A$

Hence, $(x_\alpha) \in \pi_{U_\alpha} \cap A$

$\Rightarrow \pi_{U_\alpha}$ intersects A

$\Rightarrow \bar{0} \in \bar{A}$

Now, we prove that, there is no sequence of points of A convergin of to $\bar{0}$

Let $\{a_n\}$ be a sequence of points of A

Then each point a_n is a point of $\mathbb{R}^J$ having only finitely many co-ordinates equal to zero.

Let J_n be subset of J consisting of these indices α for which 2^{th} co-ordinates of a_n is Zero

Then J_n is finite for each n

$\Rightarrow UJ_n$ is countable subset of J

But J is uncountable.

We can choose $\beta \in J$ such that $\beta \notin UJ_n$

Now, $\beta \notin UJ_n \Rightarrow B \notin J_n \; \forall n$

$\Rightarrow \beta^{th}$ co-ordinates of $a_n = 1, \forall n$

Let $U_\beta = (-1,1)$ be an open interval in $\mathbb{R}$

Let $U = \pi_\beta^{-1}(U_\beta)$ then U is open in $\mathbb{R}^J$

$\pi_\beta \bar{0} = 0 \in U_\beta$

$\Rightarrow \bar{0} \in \pi_\beta^{-1}(U_\beta) \in U$

$\therefore \bar{0} \in U$

$\therefore U$ is an neighbour hood of $\bar{0}$

Claim: $a_n \notin U \; \forall n$

Now, $\pi_\beta(a_n) = \beta^{th}$ co-ordinate of a_n

$= 1 \notin U_\beta \; \forall n$

$\therefore a_n \in \pi_\beta^{-1}(U_\beta) \; \forall n$

$\Rightarrow a_n \notin U \ \forall n$

Hence (a_n) cannot converges to $\bar{0}$ in the product topology.

$\therefore$ By sequence lemma, $\mathbb{R}^J$ is not metrizable.

UNIT - 3

CONNECTEDNESS

3.1. Connected Spaces

Definition.

Let X be a topological space. A **separation** of X is a pair U and V of disjoint nonempty open subsets of X whose union is X. A space X is said to be **connected** if there does not exist a separation of X.

Connectedness is obviously a topological property, since it is formulated entirely in terms of the collection of open sets of X. Said differently, if X is connected, so is any space homeomorphic to X.

Another way of formulating the definition of connectedness is the following:

A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

For if A is any nonempty proper subset of X which is not open and closed in X, then the sets $U = A$ and $V = X - A$ constitute a separation of X for they are open, disjoint and nonempty and their union is X.

Conversely, if U and V form a separation of X, then U is nonempty and different from X, and it is both open and closed in X.

Lemma 3.1.1.

If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exist no separation of Y.

Proof.

Suppose first that A and B form a separation of Y.

Then A is both open and closed in Y.

The closure of A in Y is the set $\bar{A} \cap Y$ (where $\bar{A}$ denote the closure of $A\ in\ X$).

Since A is closed in $Y, A = \bar{A} \cap Y$ or $\bar{A} \cap B = \emptyset$ (The limit points of A cannot lie in B)

$$[\because A \cap B = \emptyset, (\bar{A} \cap Y) \cap B = \emptyset, i.e., \bar{A} \cap (Y \cap B) = \bar{A} \cap B = \emptyset]$$

Since $\bar{A}$ is the union of A and its limit points, B contains no limit points of A. A similar argument shows that A contains no limit points of B.

Conversely, suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other.

Then $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$

$\therefore$ We conclude that $\bar{A} \cap Y = A\ and\ \bar{B} \cap Y = B$.

Thus both $A\ and\ B$ are closed in Y.

Since $A = Y - B$ and $B = Y - A$, they are open in Y as well.

$\therefore$ A and B form a separation of Y.

Example 1.

Let X denote a two – point space in the indiscrete topology. Obviously there is no separation of X and so X is connected.

Example 2.

Let $Y = [-1,0) \cup (0,1]$, Y is a subspace of the real line.

$A = [-1,0)$ and $B = (0,1]$ are disjoint nonempty subsets of Y whose union is Y, neither of which contains a limit point of the other (both are open in Y)

$\therefore$ They form a separation of Y

$\therefore$ Y is not connected.

Example 3.

Let $X = [-1,1]$ be the subspace of $\mathbb{R}$. The sets $[-1,0]$ and $(0,1]$ are disjoint nonempty whose union is X. They do not form a separation of X, because the first $[-1,0]$ is not open in X.

Alternatively, note that the first set contains a limit, 0, of the second.

∴ There is no separation of the space $[-1,1]$.

i.e., X is connected.

Example 4.

The rationales $\mathbb{Q}$ are not connected. Indeed, the only connected subspaces of $\mathbb{Q}$ are the one-point sets. If Y is a subspace of $\mathbb{Q}$ containing two points p and q, we can choose an irrational number a lying between p and q, and write Y as the union of the open sets

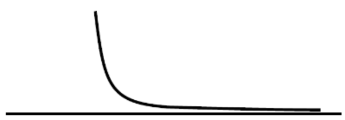
$$Y \cap (-\infty, a) \text{ and } Y \cap (a, +\infty).$$

Example 5.

Consider the following subset of the plane $\mathbb{R}$:

$$X = \{x \times y \mid y = 0\} \cup \{x \times y \mid x > 0 \text{ and } y = 1/x\}.$$

Then X is not connected; indeed, the two indicated sets form a separation of X because neither contains a limit point of the other. See Figure 3.1.1.



Lemma 3.1.2.

If the sets C and D form a separation of X, and if Y is a connected subspace of X. Then Y has to lie entirely within either C or D.

Proof.

Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y.

These two sets are disjoint and their union is Y.

If they were both nonempty, they would constitute a separation of Y.

$\therefore$ one of them is empty.

Hence Y must lie entirely in C or D.

Theorem 3.1.3.

The union of a collection of connected sets that have a point in common is connected.

Proof.

Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of connected subsets of a space X, Let p be a point in $\bigcap_{\alpha \in J} A_\alpha$.

We prove that the set $Y = \bigcup A_\alpha$ is connected.

Suppose that $Y = C \cup D$ is a separation of Y. The point $p(\in Y)$ is one of the set C or D (they are disjoint).

Suppose $p \in C$. Since the set A_α is a connected subset of Y (for each α) by the above lemma it must entirely in either C or D; it cannot lie in D, because it contains the point p of C.

Hence, $A_\alpha \subset C, \forall \alpha$.

$\therefore Y = \cup A_\alpha \subset C$ contradicting the fact that D is non empty.

This contradiction shows that Y is connected.

Theorem 3.1.4.

Let A be a connected subset of X. If $A \subset B \subset \bar{A}$, then B is also connected.

(In other words, if B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected.)

Proof.

Let A be a connected subset of X and let $A \subset B \subset \bar{A}$

Suppose that $B = C \cup D$ is a separation of B.

Since A is a connected subset of B, A must lie entirely in C or D by Lemma 3.1.2.

Suppose that $A \subset C$.

Then $\bar{A} \subset \bar{C}$

Since $\bar{C}$ and D are disjoint and since $B \subset \bar{A} \subset \bar{C}$, B cannot intersect D.

This contradicts the fact that D is a nonempty subset of B.

$\therefore$ B is connected.

Theorem 3.1.5.

The image of a connected space under a continuous map is connected.

Proof.

Let X be connected and let $f: X \to Y$ be a continuous map.

We wish to prove that the image set $Z = f(X)$ is connected.

Since the map obtained from f by restricting its image to the space is also continuous, it suffices to consider the case of a continuous subjective map $g: X \to Z$.

Suppose that $Z = A \cup B$ is a separation of Z into two disjoint nonempty sets open in Z.

Then $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint sets whose union is X; they are open in X because g is continuous and nonempty because g is surjective.

$\therefore$ They form a separation of X, contradicting the assumption that X is connected.

Hence the theorem.

Theorem 3.1.6.

The cartesian product of connected spaces is connected.

Proof.

We prove the theorem first for the product of two connected spaces X and Y.

Choose a base point $a \times b$ in the product space $X \times Y$.

Note that the horizontal slice $X \times b$ is connected, being homomorphic with X and each vertical slice $x \times Y$ is connected being homomorphic with Y.

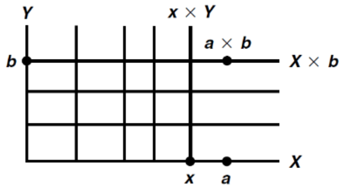


Figure 3.1.2

As a result, each “T−shaped” space $T_x = (X \times b) \cup (x \times Y)$ is connected, being the union of two connected sets that have the point $x \times b$ in common. See Figure 3.1.2.

Now, form the union $\cup_{x \in X} T_x$ of all these T −shaped spaces. This union is connected because it is the union of a collection of connected sets that have the point $a \times b$ in common. Since this union equals $X \times Y$, the space $X \times Y$ is connected.

Using induction, we see that any finite product of connected spaces $X_1 \times X_2 \times \dots \dots \times X_n$ is connected since $X_1 \times X_2 \times \dots \dots \times X_n$ is homeomorphic with $(X_1 \times X_2 \times \dots \dots \times X_{n-1}) \times X_n$.

Hence the theorem.

Result. *Next, we prove the result for an arbitrary product of connected spaces.*

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of connected spaces, and let $X = \prod_{\alpha \in J} X_\alpha$. Choose a base point $b = (b_\alpha)_{\alpha \in J}$ for X. Given any finite set $(\alpha_1, \alpha_2, \dots \dots, \alpha_n)$ of indices in J, let us define a subspace $X(\alpha_1, \dots \dots \alpha_n)$ of X.

It consists of all points $(X_\alpha)_{\alpha \in J}$. Show that $x_\alpha = b_\alpha$ for $\alpha \neq \alpha_1, \dots \dots \alpha_n$.

We assert that $X(\alpha_1, \dots \dots \alpha_n)$ is homeomorphic with finite product $X_{\alpha_1} \times \dots \dots \times X_{\alpha_n}$ and hence is connected.

Consider the mapping

$(X_{\alpha_1}, \dots \dots, X_{\alpha_n}) \to (y_\alpha)_{\alpha \in J}$ of $X_{\alpha_1} \times \dots \dots . \times X_{\alpha_n} \to X(\alpha_1, \dots \dots, \alpha_n)$, where $y_\alpha = x_\alpha$ for $\alpha = \alpha_1, \dots \dots, \alpha_n$ and $y_\alpha = b_\alpha$ for all other values of α.

This map is bijective and it arrives a basis element for $X_{\alpha_1} \times, \dots .. \times X_{\alpha_n}$ to a basis element for $X(\alpha_1, \dots \dots, \alpha_n)$.

Put, $Y = \cup \times (\alpha_1, \dots \dots \alpha_n)$, where the union is taken over all finite subsets $\{\alpha_1, \dots \dots, \alpha_n\}$ of J.

Then Y is a subspace of X.

Since the spaces $X(\alpha_1, \dots \dots \alpha_n)$ are connected and they are contain the base point $b = (b_\alpha), Y$ is connected.

But Y is not all of X. Then Y consists of all points $(x_\alpha)_{\alpha \in J}$ of X, having the property that $x_\alpha = b_\alpha$ for $\alpha \neq \alpha_1, \dots \dots, \alpha_n$.

Now we assert that the closure of Y equals all of X.

Once we prove this fact, the connectedness X follows from the theorem $(A \subset B \subset \bar{A})$

Let us take an arbitrary point (x_α) of X and an arbitrary basis element $U = \prod_{\alpha \in J} U_\alpha$ about (x_α) and prove that U intersects Y.

Each set U_α is open in X_α and $U_\alpha = X_\alpha$ except for finitely many indices, say $\alpha = \alpha_1, \dots . \alpha_n$.

Construct a point (y_α) of X by setting

$$y_\alpha = \begin{cases} x_\alpha & for\ \alpha = \alpha_1, \dots \dots, \alpha_n \\ b_\alpha\ for\ all\ other\ values\ of\ \alpha \end{cases}$$

Then (y_α) is a point of Y, because it belongs to the space $X(\alpha_1, \dots .., \alpha_n)$. (y_α) is also a point of U_1 because $y_\alpha = x_\alpha \in U_\alpha$ for $\alpha = \alpha_1, \dots \dots, \alpha_n$ and $y_\alpha = b_\alpha \in X_\alpha$ for all other values of α.

Hence U intersects Y as we desired.

Hence the result.

3.2. Components and Local Connectedness

Definition.

Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the ***components*** (or the "connected components") of X.

Result. $\sim$ is an equivalence relation on X.

Proof.

Symmetry and reflexivity of the relation are obvious.

Now, if A is a connected subspace containing x and y, and if B is a connected subspace containing y and z, then $A \cup B$ is a subspace containing x and z that is connected because A and B have the point y in common. Therefore, transitivity relation holds.

Thus $\sim$ is an equivalence relation.

Theorem 3.2.1.

The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Proof.

Since the components are equivalence classes from the equivalence relation, it is clear that the components of X are disjoint and their union is X.

Claim: each connected subspace of A of X intersects only one of them.

If A intersects the components C_1 and C_2 of X, say in points x_1 and x_2, respectively, then $x_1 \sim x_2$ by definition; this cannot happen unless $C_1 = C_2$.

To show the component C is connected.

choose a point $x_0 \in C$.

For each point $x \in C$, we know that $x_0 \sim x$, so there is a connected subspace A_x containing x_0 and x.

By the result just proved, $A_x \subset C$.

Therefore, $C = \bigcup_{x \in C} A_x$.

Since the subspaces A_x are connected and have the point x_0 in common, their union is connected.

Definition.

We define another equivalence relation on the space *X* by defining $x \sim y$ if there is a path in *X* from *x* to *y*. The equivalence classes are called the ***path components*** of *X*.

Result.

The relation ~ defined on X by $x \sim y$ if there is a path in *X* from *x* to *y.* Prove that ~ is an equivalence relation.

Proof.

First, we note that if there exists a path $f : [a, b] \rightarrow X$ from *x* to *y* whose domain is the interval [*a, b*], then there is also a path *g* from *x* to *y* having the closed interval [*c, d*] as its domain. (This follows from the fact that any two closed intervals in R are homeomorphic.)

Now the fact that $x \sim x$ for each $x \in X$ follows from the existence of the constant path $f : [a, b] \rightarrow X$ defined by the equation $f(t) = x$ for all *t*. The reflexivity holds.

From the fact that if $f: [0, 1] \rightarrow X$ is a path from *x* to *y*, then the "reverse path" $g: [0, 1] \rightarrow X$ defined by $g(t) = f(1 - t)$ is a path from *y* to *x.* This follows symmetry.

Let $f: [0, 1] \rightarrow X$ be a path from *x* to *y*, and let $g: [1, 2] \rightarrow X$ be a path from *y* to *z*.

We can "paste *f* and *g* together" to get a path $h: [0, 2] \rightarrow X$ from *x* to *z*; the path *h* will be continuous by the "pasting lemma".

Hence ~ is an equivalence relation.

Theorem 3.2.2.

The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.

Proof. Proof is similar to Theorem 3.2.1.

Note. Each component of a space X is closed in X, since the closure of a connected subspace of X is connected. If X has only finitely many components, then each component is also open in X, since its complement is a finite union of closed sets. But in general, the components of X need not be open in X.

We can say even less about the path components of X, for they need be neither open nor closed in X.

Example 1. If $\mathbb{Q}$ is the subspace of $\mathbb{R}$ consisting of the rational numbers, then each component of $\mathbb{Q}$ consists of a single point. None of the components of $\mathbb{Q}$ are open in $\mathbb{Q}$.

Example 2.

The "topologist's sine curve" $\bar{S}$ of the preceding section is a space that has a single component (since it is connected) and two path components. One path component is the curve S and the other is the vertical interval $V = 0 \times [-1, 1]$. Note that S is open in $\bar{S}$ but not closed, while V is closed but not open.

If one forms a space from $\bar{S}$ by deleting all points of V having rational second coordinate, one obtains a space that has only one component but uncountably many path components.

Definition.

A space X is said to be ***locally connected at x*** if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be ***locally connected***. Similarly, a space X is said to be ***locally path connected at x*** if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If X is locally path connected at each of its points, then it is said to be ***locally path connected***.

Example 3.

Each interval and each ray in the real line is both connected and locally connected.

The subspace $[-1, 0) \cup (0, 1]$ of $\mathbb{R}$ is not connected, but it is locally connected.

The topologist' s sine curve is connected but not locally connected. The rational $\mathbb{Q}$ are neither connected nor locally connected.

Theorem 3.2.3.

A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof.

Suppose that X is locally connected; let U be an open set in X; let C be a component of U.

If $x \in C$, we can choose a connected neighborhood V of x such that $V \subset U$.

Since V is connected, it must lie entirely in the component C of U.

Therefore, C is open in X.

Thus each component of U is open in X.

Conversely, suppose that components of open sets in X are open.

Given a point x of X and a neighborhood U of x, let C be the component of U containing x. Now C is connected; since it is open in X by hypothesis, X is locally connected at x.

Thus X is locally connected.

Theorem 3.2.4.

A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.

Proof. Proof is similar to Theorem 3.2.3.

Theorem 3.2.5.

If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Proof.

Let C be a component of X;let x be a point of C;let P be the path component of X containing x.

Since P is connected, $P \subset C$.

We wish to show that if X is locally path connected, $P = C$.

Suppose that $P \subsetneq C$.

Let Q denote the union of all the path components of X that are different from P and intersect C;

Then each of them necessarily lies in C, so that

$$C = P \cup Q.$$

Because X is locally path connected, each path component of X is open in X.

Therefore, P (which is a path component) and Q (which is a union of path components) are open in X, so they constitute a separation of C.

This contradicts the fact that C is connected.

Thus $P = C$.

UNIT – 4

COMPACTNESS

4.1. Compact Spaces

Definition.

A collection $\mathcal{A}$ of subsets of a topological space X is said to **cover** X or to be a **covering** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

Definition.

A topological space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Example 1.

The real line $\mathbb{R}$ is not compact. $\mathcal{A} = \{(n, n+2)/n \epsilon \mathbb{Z}\}$ is an open covering of $\mathbb{R}$. But it contains no finite sub collection that covers $\mathbb{R}$.

Example 2.

Let $X = \{0\} \cup \{\frac{1}{n}/n \in \mathbb{Z}_+\}$. This is a subspace of $\mathbb{R}$.

Given an open covering $\mathcal{A}$ of X, there is an element U of A containing 0. The set U contains all but finitely many of the points $\frac{1}{n}$. Choose for each point of X not in U, an element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$, along with the element U, is a finite collection of $\mathcal{A}$, that covers X.

$\therefore X$ is compact.

Definition.

If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to ***cover Y*** if the union of its elements *contains* Y.

Lemmas 4.1.1.

Let Y be a subspace of X. Then Y is compact iff every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof.

Given Y be a subspace of X.

Assume that Y is compact.

Let $\mathcal{A} = \{A_\alpha / \alpha \in J\}$ be a covering of Y, where A_α is open in X.

To prove $Y \subseteq \bigcup_{i=1}^{n} A_{\alpha_i}$

Since A_α is open in X

$\Rightarrow A_\alpha \cap Y$ is open in Y.

$\Rightarrow \cup (A_\alpha \cap Y) = (\cup A_\alpha) \cap Y$

$= Y \cap Y$

$\Rightarrow \cup (A_\alpha \cap Y) = Y$

$\therefore \{A_\alpha \cap Y / \alpha \in J\}$ is an covering of Y.

Since Y is compact.

$\therefore$ The above open cover has a finite subcover $\{A_{\alpha_1} \cap Y, A_{\alpha_2} \cap Y, \dots \dots, A_{\alpha_n} \cap Y\}$

$\Rightarrow \bigcup_{i=1}^{n}(A_{\alpha_i} \cap Y) = Y$

$\Rightarrow \left(\bigcup_{i=1}^{n} A_{\alpha_i}\right) \cap Y = Y$

$\Rightarrow Y \subseteq \bigcup_{i=1}^{n}(A_{\alpha_i})$

i.e., every open covering of Y by sets open in X contains a finite subcollection covering Y.

Conversely, assume that every open covering of Y by sets open in X contains a finite subcollection covering Y.

To prove: Y is compact.

Let $A' = \{A'_\alpha / \alpha \in J\}$ be an open covering of Y, where A'_α is open in Y.

$\therefore \cup A'_\alpha = Y,\ \ where\ \alpha \in J$

Since A'_α is open in Y.

$\therefore A'_\alpha = A_\alpha \cap Y$, where A_α is open in X.

$\Rightarrow Y = \cup (A_\alpha \cap Y)$

$\Rightarrow Y = (\cup A_\alpha) \cap Y$

$\Rightarrow Y \subset \cup (A_\alpha)$

i.e., The set $\{A_\alpha / \alpha \in J\}$ is a open covering of Y by sets open in X.

By assumption, this has a finite subcollection that covers Y.

i.e. $Y \subseteq \bigcup_{i=1}^{n} A_{\alpha_i}$

$\therefore Y = \left(\bigcup_{i=1}^{n} A_{\alpha_i}\right) \cap Y$

$\therefore Y \subseteq \bigcup_{i=1}^{n}(A_{\alpha_i})$

i.e., $Y \subseteq \bigcup_{i=1}^{n} A'_{\alpha_i}$

A' has a finite subcollection $\{A'_{\alpha_1}, \dots \dots, A'_{\alpha_n}\}$ that covers Y.

Hence Y is compact.

Theorem 4.1.2.

Every closed subspace of a compact space is compact.

Proof.

Let X be a compact space and let Y be a closed subspace of X.

To prove: Y is compact.

Enough to prove that every covering of Y by sets open in X contains a finite subcollection covering Y.

Let $\mathcal{A}$ be a covering of Y by sets open in X.

Since Y is closed in X, X\Y is open in X.

$\therefore \mathcal{B} = \mathcal{A} \cup (X \backslash Y)$ is an open cover of X.

Since X is compact.

$\Rightarrow \mathcal{B}$ contains a finite subcollection covering X.

If this subcollection contains the set $X \backslash Y$, discard $X - Y$, otherwise, leave the subcollection alone.

$\therefore$ The resulting subcollection is a finite subcollection of $\mathcal{A}$ that covers Y.

Hence Y is compact.

Theorem 4.1.3.

Every compact subspace of a Haussdorff space is closed.

Proof.

Lemma 4.1.4: If Y is a compact subspace of Hausdorff space X and $x_0 \notin Y$. Then there exist a disjoint open sets U and V of X containing x_0 and y respectively.

Proof of lemma.

Given Y is a compact subspace X and $x_0 \notin Y$.

$\Rightarrow x_0 \in X \backslash Y$

Let $y \in Y \ \Rightarrow x_0 \neq y$

i.e, x_0 and y are two distinct points in X.

since X is Hausdorff.

$\Rightarrow \exists$ two open sets U_y, V_y of x_0 and y respectively. Show that $U_y \cap V_y \neq 0$.

$\therefore$ The collection $\{V_y / y \in Y\}$ is a covering of Y by sets open in X.

Since Y is compact.

$\therefore$ The above open cover has a finite subcollection, say $\{V_{y_1}, V_{y_2}, \dots \dots, V_{y_n}\}$, that covers Y.

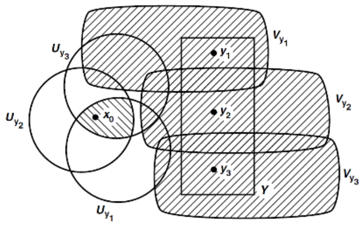
i.e., $Y \subseteq V_{y_1} \cup V_{y_2} \cup \dots \dots \cup V_{y_n}$

let $V = V_{y_1} \cup \dots \dots \cup V_{y_n}$, which is an open set containing Y.

Taking $U_{y_1}, U_{y_2}, \dots \dots, U_{y_n}$ be the corresponding neighbourhoods of x_0.

Let $U = U_{y_1} \cap \dots \dots \cup U_{y_n}$, which is an open set and $x_0 \in U$.

i.e., we have found out two open sets U and V such that $x_0 \in U$ and $Y \subset V$.



Now to prove $U \cap V = \emptyset$.

Suppose $U \cap V \neq \emptyset, \exists$ at least one element, say $x \in U \cap V$,

$\Rightarrow x \in U$ and $x \in V$

$\Rightarrow x \in U_{y_1} \cap U_{y_2} \cap \dots \dots \cap U_{y_n}$ and $x \in V_{y_1} \cup \dots \dots \cup V_{y_n}$

$\Rightarrow x \in U_{y_i}, \forall\, i = 1, \dots \dots n, x \in V_{y_j}$ for some j

$\Rightarrow x \in U_{y_i} \cap V_{y_j}$

$\Rightarrow \Leftarrow \left(Since\ U_{y_i} and\ V_{y_j}\ are\ disjoint\right)$

Hence $U \cap V = \emptyset$.

Proof of the theorem.

Let X be a housedorff space and let Y be a compact subspace of X.

To prove Y is closed in X.

i.e., to prove Y^C is open in X.

i.e., to prove $X \backslash Y$ is open in X

Let $x_0 \in X\backslash Y$

By previous theorem, $\exists$ disjoint open sets U and V show that $x_0 \in U$ and $y \subseteq V$ and $U \cap V = \emptyset$.

Now, $U \cap V = \emptyset$(since $U \subseteq V^C \subseteq Y^C = X\backslash Y$)

$\therefore$ For each $x_0 \in X\backslash Y, \exists$ and open set U containing x_0 show that $x_0 \in U \subseteq X\backslash Y$.

$\Rightarrow X\backslash Y$ is open in X.

i.e., Y is closed in X.

Example 5.

Once we prove that the interval [*a, b*] in R is compact, it follows from Theorem 4.1.2 that any closed subspace of [*a, b*] is compact. On the other hand, it follows from Theorem 4.1.3 that the intervals *(a, b*] and *(a, b)* in R cannot be compact (which we knew already) because they are not closed in the Hausdorff space R.

Example 6.

One needs the Hausdorff condition in the hypothesis of Theorem 4.1.3. Consider, for example, the finite complement topology on the real line. The only proper subsets of R that are closed in this topology are the finite sets. But *every* subset of R is compact in this topology.

Theorem 4.1.5.

The image of a compact space under a continuous map is compact.

Proof.

Let X be a compact space and Y be a subspace of X.

Let $f: X \rightarrow Y$ be a continuous map.

To prove**:** $f(X)$ is compact.

Let $\mathcal{A} = \{A_\alpha / \alpha \in J\}$ be an open cover for $f(X)$ by sets open in Y.

Since $A_\alpha's$ are open in Y.

$\Rightarrow f^{-1}(A_\alpha)$ is opne in $X[\because f \text{ } is \text{ } continuous]$

$\Rightarrow \{f^{-1}(A_\alpha)/\alpha \in J\}$ is an open cover for X.

Since X is compact, the above open cover has a finite sub collection that covers X.

i.e., $X \subseteq \bigcup_{i=1}^{n} f^{-1}(A_{\alpha_i})$

$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} A_{\alpha_i}$

i.e., $\{A_{\alpha_1}, \dots \dots, A_{\alpha_n}\}$ is a finite sub collection of $\mathcal{A}$ that covers $f(X)$ and hence $f(X)$ is compact.

Thus the continuous image of a compact space is compact.

Theorem 4.1.6.

Let $f: X \to Y$ be a bijective continuous function. If X is compact anf Y is Hausdarff space. Then f is a homeomorphism.

Proof.

Given $f: X \to Y$ is a bijective continuous map and let X be compact and Y be hausdorff.

To prove f is homeomorphism.

i.e., To prove f^{-1} is contuinuous.

In order to prove, if A is closed in $X \Rightarrow (f^{-1})^{-1}(A)$ is closed in Y.

$\because$ [f is continuous iff for every close set B in Y, the set $f^{-1}(B)$ is closed in X]

s To prove $f(A)$ is closed in Y.

Since A is closed in X.

$\Rightarrow A$ is compact [$\because$every closed subspace of compact space is compact]

$\Rightarrow f(A)$ is compact [by previous theorem]

$\Longrightarrow f(A)$ is closed [∵every compact subspace of a hausdorff space is closed]

$\Longrightarrow f^{-1}$ is continuous.

∴ f is homeomorphism.

Theorem 4.1.7.

The product of finitely many compact spaces is compact

Proof.

Let $X_1, X_2, \ldots, X_n$ be compact spaces.

To prove $X_1 \times X_2 \times \ldots \times X_n$ is compact.

First, we shall prove that the product of two compact space is compact. Then the theorem follows by induction for any finite product.

Before proving this theorem, let us prove the Tube lemma.

Lemma 4.1.8.(Tuba Lemma).

Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is the neighbourhood of x_0 in X.

Proof of the Lemma.

Suppose that we are given two spaces X and Y, with Y is a compact space.

Suppose that $x_0 \in X$ and N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of X× Y.

To prove N contains the tube $W \times Y$ about $x_0 \times Y$, where W is the neighbourhood of x_0 in X.

Since N is open in $X \times Y$ containing $x_0 \times Y$

$\Longrightarrow \exists$ a basis element $U \times V$ in $X \times Y$ Such that $x_0 \times Y \in U \times V \subseteq N$ [∵ U is open in X, V is open in Y]

∴ The collection $\mathcal{A} = \{U \times V / U\ is$ open in X and V is open in $Y\}$ is the open cover for $x_0 \times Y$ by sets open in $X \times Y$.

Since $x_0 \times Y$ is homeomorphic with Y.

$\therefore x_0 \times Y$ is compact.

$\therefore \mathcal{A}$ has a finite subcollection contains $x_0 \times Y$.

i.e., $x_0 \times Y \subseteq U_1 \times V_1 \cup U_2 \times V_2 \cup \ldots \ldots . . \cup U_n \times V_n$ -------------(1)

Let $W = U_1 \cap U_2 \cap \ldots \ldots \cap U_n$

Since each U_i is open in X, W is open in X.

Since $x_0 \in U_i \; \forall i = 1 \ldots \ldots n$ and $x_0 \in W$ also.

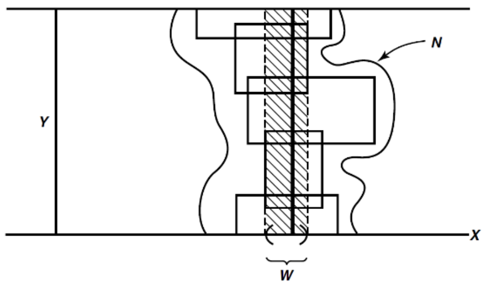


Figure 4.1.2

$\Longrightarrow x_0 \times Y \in W \times Y$

To prove $W \times Y \subseteq N$

Let $x \times y \in W \times Y$

$\Longrightarrow x \in W \; and \; y \in Y$

$\Longrightarrow x \in U_i \; \forall i \; and \; y \in V_j$ for some j

$\therefore x \times y \in U_i \times V_j$ for some j

$\Longrightarrow x \times y \in W \times Y$

$\Longrightarrow x \times y \in N$ ($\therefore$ all the sets $U_i \times V_j$ lie in N)

$\therefore W \times Y \subseteq N$

Hence the Lemma.

Proof of the theorem.

We shall prove that the product of finite two compact space is compact and the theorem follows by induction on any finite product of finite.

Let $X \; and \; Y$ be two compact spaces.

To prove $X \times Y$ is compact.

Let $\mathcal{A} = \{A_\alpha / A_\alpha{}'s$ are open in $X\}$ be an open covering for $X \times Y$.

Let $x_0 \in X$

Consider the slice $x_0 \times Y$. Clearly $\mathcal{A}$ is a covering of $x_0 \times Y$ by sets open in $X \times Y$.

Since $x_0 \times Y$ is homeomorphic with Y and Y is compact.

$\Longrightarrow x_0 \times Y$ is compact.

$\therefore \mathcal{A}$ has a finite subcollection $A_{\alpha_1}, A_{\alpha_2}, \dots \dots, A_{\alpha_n}$ such that $x_0 \times Y$ is contained in $A_{\alpha_1} \cup A_{\alpha_2} \cup \dots \dots \dots \cup A_{\alpha_n} = N$.

Each A_{α_i} is open in $X \times Y$ and N is an open in $X \times Y$ containing the slice $x_0 \times Y$.

$\therefore$ by Tube lemma, $\exists$ a Tube $W \times Y$ about $x_0 \times Y$ such that $W \times Y \subseteq N$, where W is a neighbourhood of x_0.

$\therefore$ For each $x \in X$ we can choose a neighbourhood W_x of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$.

Consider the collection $\mathcal{A}' = \{W_x / x \in X\}$ is an open covering of X.

Since X is compact.

$\Longrightarrow \exists$ a finite subcollection $W_{x_1}, W_{x_2} \dots \dots, W_{x_n}$ of A' show that $X = W_{x_1} \cup W_{x_2} \cup \dots \dots \cup W_{x_n}$.

Then $X \times Y = \left(W_{x_1} \times Y\right) \cup \left(W_{x_2} \times Y\right) \cup \dots \dots \cup \left(W_{x_n} \times Y\right)$

i.e., The collection $W_{x_1} \times Y, W_{x_2} \times Y, \dots \dots, W_{x_n} \times Y$ forms a covering of $X \times Y$ -----(2)

from (1) and (2),we conclude that $X \times Y$ is covered by finitely many elements of $\mathcal{A}$.

$\Longrightarrow X \times Y$ is compact.

$\therefore$ The product of two compact spaces is compact.

$\therefore$ By using induction method, we get product sof finitely many compact spaces is compact.

Definition.

A collection $\mathcal{C}$ of subset of X is said to have a **finite intersection property** of for every finite sub collection $C_1, C_2, \dots \dots, C_n$ of $\mathcal{C}$ such that $\bigcap_{i=1}^{n} C_i \neq \emptyset$.

Theorem 4.1.9.

Let X be a topological space. Then X is compact iff for every collection $\mathcal{C}$ of closed sets in X having finite intersection property, the intersection $\bigcap_{c\in\mathcal{C}} C$ of all elements of $\mathcal{C}$ is nonempty.

Proof.

Assume that X is compact and $\mathcal{C}$ is a collection of closed sets in X satisfying the finite intersection property.

To prove $\bigcap_{c\in\mathcal{C}} C \neq \emptyset$

Suppose not, i.e., $\bigcap_{c\in\mathcal{C}} C \neq \emptyset$

$\Rightarrow (\bigcap_{C\in\mathcal{C}} C)^C = \emptyset^C = X$

$\Rightarrow X\backslash \cap C = X$

$\bigcup_{C\in\mathcal{C}}(X\backslash C) = X$

$\therefore \{X\backslash C / C \in \mathcal{C}\}$ is open cover for X.

Since X is compact, This open cover has a finitely subcover $X\backslash C_1, X\backslash C_2, \dots \dots, X\backslash C_n$

i.e.,$\bigcup_{i=1}^{n}(X\backslash C_i) = X$

$\Rightarrow X\backslash \bigcap_{i=1}^{n} C_i = X$

Taking complement,

$\therefore \bigcap_{i=1}^{n} C_i = \emptyset$

$\Rightarrow\Leftarrow$ [$\because$ every collection of closed set has a finite intersection property]

Hence $\bigcap_{c\in\mathcal{C}} C \neq \emptyset$

Conversely, assume that for every collection $\mathcal{C}$ of closed sets in X satisfying finite intersection property, $\bigcap_{c\in\mathcal{C}} C \neq \emptyset$

To prove X is compact.

Let $\{A_\alpha / \alpha \in J\}$ be an open cover for X.

To prove this has a finite subcover.

Suppose it does not have a finite subcover.

Since $\{A_\alpha/\alpha \in J\}$ is an open in X.

$\Rightarrow X\backslash A_\alpha$ is closed set in X.

$\mathcal{C} = \{X\backslash A_\alpha/\alpha \in J\}$ is a collection of closed sets.

Since $\{A_\alpha/\alpha \in J\}$ is an open cover for X.

$$\Rightarrow \bigcup_{\alpha\in J} A_\alpha = X$$

$$\Rightarrow \left(X\backslash \bigcup_{\alpha\in J} A_\alpha\right) = \emptyset$$

$$\Rightarrow \bigcap_{\alpha\in J}(X\backslash A_\alpha) = \emptyset$$

∴ We conclude that $\mathcal{C} = \{X\backslash A_\alpha/\alpha \in J\}$ is a collection of closed sets having empty intersection.

∴ $\mathcal{C}$ does not satisfy the finite intersection property.

i.e., ∃ a finite sets in $\mathcal{C}$, namely $X\backslash A_{\alpha_1}, X\backslash A_{\alpha_2}, \dots \dots, X\backslash A_{\alpha_n}$, show that

$\bigcap_{i=1}^{n}\left(X\backslash A_{\alpha_i}\right) = \emptyset$

$\Rightarrow X\backslash \bigcup_{i=1}^{n} A_{\alpha_i} = \emptyset$

$\Rightarrow \bigcup_{i=1}^{n} A_{\alpha_i} = X$

⇒ A has a finite subcover.

The open cover what we have chosen has a finite subcover.

∴ Our assumption is wrong. i.e, every open cover for X has a finite subcover and hence X is compact.

Corollary.

The space X is compact iff for every collection $\mathcal{A}$ of subset of X satisfying finite intersection property $\bigcap_{A\in\mathcal{A}} \neq \emptyset$.

Proof.

Assume that X is compact.

Let $\mathcal{A} = \{A_\alpha / \alpha \in J\}$ be a collection of subsets of X having a finite intersection property.

To prove $\bigcap_{i=1}^{n} A_{\alpha_i} \neq \emptyset$

Consider $\mathcal{A}' = \{\overline{A_\alpha} / \alpha \in J\}$

Since $\mathcal{A}$ satisfy finite intersection property.

i.e., $\bigcap_{i=1}^{n} A_{\alpha_i} \neq \emptyset$

$\Rightarrow \bigcap_{i=1}^{n} A_{\alpha_i} \subseteq \bigcap_{i=1}^{n} \bar{A}_{\alpha_i} \neq \emptyset$

$\Rightarrow \bigcap_{i=1}^{n} \bar{A}_{\alpha_i} \neq \emptyset$

$\Rightarrow A'$ satisfies a finite intersection property.

$\therefore$ X is compact $\Rightarrow \bigcap_{A \in \mathcal{A}} \bar{A} \neq \emptyset$

Conversely, assume that for every collection $\mathcal{A}$ of subsets of X satisfying finite intersection property $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$

To prove X is compact.

Let $\mathcal{C}$ be a collection of closed sets in X satisfying finite intersection condition.

$\therefore$ Our assumption, every collection $\mathcal{A}$ of subsets X satisfying finite intersection condition. We have,

$\bigcap_{A \in \mathcal{A}} \bar{A} \neq \emptyset$

$\therefore \bigcap_{c \in \mathcal{C}} \bar{C} \neq \emptyset$

Since $\mathcal{C}$ has closed sets.

$\Rightarrow C = \bar{C}$

i.e., $\bigcap_{c \in \mathcal{C}} C \neq \emptyset$

$\therefore$ By previous theorem, we get X is compact.

4.2.Limit Point Compactness

Definition.

A space X is said to be ***limit point compact*** if every infinite subset of X has a limit point.

Theorem 4.2.1.

Compactness implies limit point compactness, but not conversely.

Proof.

Let X be a compact space. Given a subset A of X.

we wish to prove that if A is infinite, then A has a limit point.

We prove the contrapositive–if A has no limit point, then A must be finite.

So, suppose A has no limit point. Then A contains all its limit points, so that A is closed.

Furthermore, for each $a \in A$ we can choose a neighborhood U_a of a such that U_a intersects A in the point a alone. The space X is covered by the open set $X - A$ and the open sets U_a; being compact, it can be covered by finitely many of these sets.

Since $X - A$ does not intersect A, and each set Ua contains only one point of A, the set A must be finite.

Example 1.

Let Y consist of two points; give Y the topology consisting of Y and the empty set. Then the space $X = Z+ \times Y$ is limit point compact, for *every* nonempty subset of X has a limit point. It is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ has no finite subcollection covering X.

Example 2.

Consider the minimal uncountable wellordered set S_ω , in the order topology.

The space S_ω is not compact, since it has no largest element.

However, it is limit point compact: Let A be an infinite subset of S_ω .

Choose a subset B of A that is countably infinite.

Being countable, the set B has an upper bound b in S_ω ; then B is a subset of the interval $[a_0, b]$ of S_ω , where a_0 is the smallest element of S_ω .

Since S_ω has the least upper bound property, the interval $[a_0, b]$ is compact.

By the preceding theorem, B has a limit point x in $[a_0, b]$.

The point x is also a limit point of A.

Thus S_ω is limit point compact.

Definition.

Let X be a topological space. If (x_n) is a sequence of points of X, and if $n_1 < n_2 < \cdots < n_i < \cdots$ is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a ***subsequence*** of the sequence (x_n). The space X is said to be ***sequentially compact*** if every sequence of points of X has a convergent subsequence.

Theorem 4.2.2.

Let X be a metrizable space. Then the following are equivalent:

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Proof.

We have already proved that *(1) ⇒ (2)*.

To show that *(2) ⇒ (3)*

Assume that X is limit point compact.

Given a sequence (x_n) of points of X, consider the set $A = \{x_n \mid n \in Z_+\}$.

If the set A is finite, then there is a point x such that $x = x_n$ for infinitely many values of n.

In this case, the sequence (x_n) has a subsequence that is constant, and therefore converges trivially.

On the other hand, if A is infinite, then A has a limit point x. We define a subsequence of *(xn)* converging to x as follows: First choose n_1 so that

$$x_{n_1} \in B(x, 1).$$

Then suppose that the positive integer n_{i-1} is given. Because the ball $B(x, 1/i)$ intersects A in infinitely many points, we can choose an index $n_i > n_{i-1}$ such that

$$x_{n_i} \in B(x, 1/i).$$

Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ converges to x.

Finally, we show that *(3)* ⇒ *(1)*.

First, we show that if X is sequentially compact, then the Lebesgue number lemma holds for X.

Let A be an open covering of X. We assume that there is no $\delta > 0$ such that each set of diameter less than δ has an element of A containing it, and derive a contradiction.

Our assumption implies in particular that for each positive integer n, there exists a set of diameter less than $1/n$ that is not contained in any element of A; let C_n be such a set. Choose a point $x_n \in C_n$, for each n. By hypothesis, some subsequence (x_{n_i}) of the sequence (x_n) converges, say to the point a. Now$a$ belongs to some element A of the collection A; because A is open, we may choose an $\epsilon > 0$ such that $B(a, \epsilon) \subset A$. If i is large enough that $1/n_i < \epsilon/2$, then the set C_{n_i} lies in the $\epsilon/2$-neighborhood of x_{n_i} ; if i is also chosen large enough that $d(x_{n_i}, a) < \epsilon/2$, then C_{n_i} lies in the ϵ-neighborhood of a. But this means that $C_{n_i} \subset A$, contrary to hypothesis.

Second, we show that if X is sequentially compact, then given $\epsilon > 0$, there exists a finite covering of X by open ϵ-balls. Once again, we proceed by contradiction.

Assume that there exists an $\epsilon > 0$ such that X cannot be covered by finitely many ϵ-balls. Construct a sequence of points x_n of X as follows: First, choose x_1 to be any point of X.

Noting that the ball $B(x_1, \epsilon)$ is not all of X (otherwise X could be covered by a single ϵ-ball), choose x_2 to be a point of X not in $B(x_1, \epsilon)$. In general, given $x_1, \ldots, x_n$, choose x_{n+1} to be a point not in the union

$$B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon),$$

using the fact that these balls do not cover X. Note that by construction $d(x_{n+1}, x_i) \geq \epsilon \; for \; i = 1, \ldots, n$. Therefore, the sequence (x_n) can have no convergent subsequence; in fact, any ball of radius $\epsilon/2$ can contain x_n for at most *one* value of n.

Finally, we show that if X is sequentially compact, then X is compact. Let A be an open covering of X. Because X is sequentially compact, the open covering A has a Lebesgue number δ. Let $\epsilon = \delta/3$; use sequential compactness of X to find a finite covering of X by openϵ-balls. Each of these balls has diameter at most $2\delta/3$, so it lies in an element of $\mathcal{A}$. Choosing one such element of A for each of these ϵ-balls, we obtain a finite subcollection of $\mathcal{A}s$ that covers X.

4.3.Local Compactness

Definition.

A space X is said to be ***locally compact at x*** if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said simply to be ***locally compact***.

Note. A compact space is automatically locally compact.

Example 1.

The real line $\mathbb{R}$ is locally compact. The point x lies in some interval *(a, b)*, which in turn is contained in the compact subspace [*a, b*]. The subspace $\mathbb{Q}$ of rational numbers is not locally compact.

Example 2.

The space $\mathbb{R}^n$ is locally compact; the point x lies in some basis element $(a_1, b_1) \times \cdots \times (a_n, b_n)$, which in turn lies in the compact subspace $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

The space $\mathbb{R}^\omega$ is not locally compact; *none* of its basis elements are contained in compact

subspaces. For if

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \cdots \times \mathbb{R} \times \cdots$$

were contained in a compact subspace, then its closure

$$\bar{B} = [a_1, b_1] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \cdots$$

would be compact, which it is not.

Example 3.

Every simply ordered set *X* having the least upper bound property is locally compact: Given a basis element for *X*, it is contained in a closed interval in *X*, which is compact.

Theorem 4.3.1.

Let *X* be a space. Then *X* is locally compact Hausdorff if and only if there exists a space *Y* satisfying the following conditions:

(1) *X* is a subspace of *Y*.

(2) The set $Y - X$ consists of a single point.

(3) *Y* is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of *Y* with Y' that equals the identity map on *X*.

Proof.

Step 1. We first verify uniqueness. Let Y and Y' be two spaces satisfying these conditions. Define $h : Y \to Y'$ by letting h map the single point p of $Y - X$ to the point q of $Y' - X$, and letting h equal the identity on X. We show that if U is open in Y, then $h(U)$ is open in Y'. Then Symmetry implies that h is a homeomorphism.

First, consider the case where U does not contain p. Then $h(U) = U$. Since U is open in Y and is contained in X, it is open in X. Because X is open in Y', the set U is also open in Y', as desired.

Second, suppose that U contains p. Since $C = Y - U$ is closed in Y, it is compact as a subspace of Y. Because C is contained in X, it is a compact subspace of X. Then because X is a subspace of Y', the space C is also a compact subspace of Y'. Because Y' is Hausdorff, C is closed in Y', so that $h(U) = Y' - C$ is open in Y', as desired.

Step 2. Now we suppose X is locally compact Hausdorff and construct the space Y. Step 1 gives us an idea how to proceed. Let us take some object that is not a point of X, denote it by the symbol ∞ for convenience, and adjoin it to X, forming the set $Y = X \cup \{\infty\}$. Topologize Y by defining the collection of open sets of Y to consist of (1) all sets U that are open in X, and (2) all sets of the form $Y - C$, where C is a compact subspace of X.

We need to check that this collection is, in fact, a topology on Y. The empty set is a set of type (1), and the space Y is a set of type (2). Checking that the intersection of two open sets is open involves three cases:

$U_1 \cap U_2$ is of type (1).

$(Y - C_1) \cap (Y - C_2) = Y - (C_1 \cup C_2)$ is of type (2).

$U_1 \cap (Y - C_1) = U_1 \cap (X - C_1)$ is of type (1),

because C_1 is closed in X. Similarly, one checks that the union of any collection of open sets is open:

$\bigcup U_\alpha = U$ is of type (1).

$\bigcup (Y - C_\beta) = Y - \left(\bigcap C_\beta\right) = Y - C$ is of type (2).

$(\bigcup U_\alpha) \cup (\bigcup (Y - C_\beta)) = U \cup (Y - C) = Y - (C - U)$, which is of type (2) because $C - U$ is a closed subspace of C and therefore compact. Now we show that X is a subspace of Y. Given any open set of Y, we show its intersection with X is open in X. If U is of type (1), then $U \cap X = U$; if $Y - C$ is of type (2), then $(Y - C) \cap X = X - C$; both of these sets are open in X. Conversely, any set open in X is a set of type (1) and therefore open in Y by definition.

To show that Y is compact, let $\mathcal{A}$ be an open covering of Y. The collection A must contain an open set of type (2), say $Y - C$, since none of the open sets of type (1) contain the point ∞. Take all the members of A different from $Y - C$ and intersect them with X; they form a collection of open sets of X covering C. Because C is compact, finitely many of them cover C; the corresponding finite collection of elements of $\mathcal{A}$ will, along with the element $Y - C$, cover all of Y.

To show that Y is Hausdorff, let x and y be two points of Y. If both of them lie in X, there are disjoint sets U and V open in X containing them, respectively. On the other hand, if $x \in X$ and $y = \infty$, we can choose a compact set C in X containing a neighborhood U of x. Then U and $Y - C$ are disjoint neighborhoods of x and ∞, respectively, in Y.

Step 3. Finally, we prove the converse. Suppose a space Y satisfying conditions (1)-(3) exists. Then X is Hausdorff because it is a subspace of the Hausdorff space Y. Given $x \in X$, we show X is locally compact at x. Choose disjoint open sets U and V of Y containing x and the single point of $Y - X$, respectively. Then the set $C = Y - V$ is closed in Y, so it is a compact subspace of Y. Since C lies in X, it is also compact as a subspace of X; it contains the neighborhood U of x.

If X itself should happen to be compact, then the space Y of the preceding theorem is not very interesting, for it is obtained from X by adjoining a single isolated point. However, if X is not compact, then the point of $Y - X$ is a limit point of X, so that. $\bar{X} = Y$.

Definition.

If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a ***compactification*** of X. If $Y - X$ equals a single point, then Y is called the ***one-point compactification*** of X.

Example 4.

The one-point compactification of the real line $\mathbb{R}$ is homeomorphic with the circle. Similarly, the one-point compactification of $\mathbb{R}^2$ is homeomorphic to the sphere S^2. If $\mathbb{R}^2$ is looked at as the space $\mathbb{C}$ of complex numbers, then $\mathbb{C} \cup \{\infty\}$ is called the *Riemann sphere*, or the

Theorem 4.3.2.

Let X be a Hausdorff space. Then X is locally compact if and only if given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

Proof.

Clearly this new formulation implies local compactness; the set $C = \bar{V}$ is the desired compact set containing a neighborhood of x. To prove the converse, suppose X is locally compact; let x be a point of X and let U be a neighborhood of x. Take the one-point compactification Y of X, and let C be the set $Y - U$. Then C is closed in Y, so that C is a compact subspace of Y. Apply Lemma 26.4 to choose disjoint open sets V and W containing x and C, respectively. Then the closure $\bar{V}$ of V in Y is compact; furthermore, $\bar{V}$ is disjoint from C, so that $\bar{V} \subset U$, as desired.

Corollary 4.3.3.

Let X be locally compact Hausdorff; let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

Proof.

Suppose that A is closed in X. Given $x \in A$, let C be a compact subspace of X containing the neighborhood U of x in X. Then $C \cap A$ is closed in C and thus compact, and it contains the neighborhood $U \cap A$ of x in A.

Suppose now that A is open in X. Given $x \in A$, we apply the preceding theorem to choose a neighborhood V of x in X such that $\bar{V}$ is compact and $\bar{V} \subset A$. Then $C = \bar{V}$ is a compact subspace of A containing the neighborhood V of x in A.

Corollary 4.3.4.

A space *X* is homeomorphic to an open subspace of a compact Hausdorff space if and only if *X* is locally compact Hausdorff.

Proof. This follows from Theorem 4.3.1 and Corollary 4.3.3.

UNIT – 5

COUNTABILITY AND SEPERATION AXIOMS

5.1. The Countability Axioms

Definition.

A space X is said to have a ***countable basis at x*** if there is a countable collection $\mathcal{B}$ of neighbourhood of x such that each neihbourhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the ***first countability axiom***, or to be ***first-countable.***

Theorem 5.1.1.

Let X be a topological space.

(a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is first-countable.

(b) Let $f: X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first countable.

Proof.

(a) Suppose $x \in \bar{A}$. Since X is first countable, there exists a countable basis say U_n at x.

Let $A_n = U_1 \cap U_2 \cap \ldots \cap U_n$ for $n = 1, 2, \ldots$

Then $\{A_n\}$ is a countable collection of neighbourhood of x and $A_1 \supset A_2 \supset \ldots \supset A_n \supset A_{n+1} \supset \ldots$

Claim: $\{A_n\}$ is a countable basis at x.

Let U be a neihbourhood of x. Since U_n is a countable basis at x, there exists U_k in $\{U_n\}$ such that $U_k \subset U$.

Also, $A_k \subset U_k$. Therefore, we have $A_k \subset U_k \subset U$.

That is $x \in A_k \subset U$.

Therefore, $\{A_n\}$ is a countable basis at x.

Now, for any n, $A_n \cap A \neq \emptyset$.

Choose $x_n \in A_n \cap A$ for $n = 1, 2, \ldots$

Now, we have a sequence (x_n) in A such that $x_n \in A_n$ for $n = 1, 2, \ldots.$

Claim: $(x_n) \rightarrow x$.

Let V be a neigbourhood of x.

Since $\{A_n\}$ is a countable basis at x, there exists x such that $A_N \subset V$.

Also, $A_n \subset A_N \; \forall \, n \geq N$.

Therefore, xn ∈ An ⊂ V

⇒ xn ∈ V ∀ n ≥ N.

Therefore, $(x_n) \rightarrow x$.

Conversely, suppose there exists a sequence (x_n) in A such that $(x_n) \rightarrow x$.

To prove $x \in \bar{A}$

Suppose there exists a sequence of points in A converging to x.

Let W be a neighbourhood of x.

Since $(x_n) \rightarrow x$ and W is a neighbourhood of x, there exists a positive integer N such that $x_n \in W, \forall \, n \geq N$.

$\Rightarrow W \cap A \neq \emptyset$.

Therefore, $x \in \bar{A}$.

Suppose $f: X \rightarrow Y$ is continuous.

To prove $(f(x_n)) \rightarrow f(x)$ where $(x_n) \rightarrow x$.

Let $(x_n) \rightarrow x$. Let V be the neigbourhood of $f(x)$.

$\Rightarrow f^{-1}(V)$ is the neigbourhood of x.

Since $(x_n) \to x$, there exists a positive integer N such that $x_n \in f^{-1}(V), \forall\, n \geq N$

$\Rightarrow f(x_n) \in V\ \forall\, n \geq N$.

Therefore, $(f(x_n)) \to f(x)$.

Conversely, suppose that $(f(x_n)) \to f(x)$ whenever $(x_n) \to x$.

To prove f is continuous.

It is enough to prove $f(\bar{A}) \subset \overline{f(A)}$ for any subset A of X.

Let $y \in \overline{f(A)}$. Then $y = f(x)$ for some $x \in A$.

Now, $x \in \bar{A}$. By (a), there exists a sequence (x_n) in A such that $(x_n) \to x$.

By hypothesis, $(f(xn)) \to f(x)$.

Then by (a), $f(x) \in \overline{f(A)}$

$\Rightarrow y \in \overline{f(A)}$.

Therefore, $f(\bar{A}) \subset \overline{f(A)}$.

Hence f is continuous.

Definition.

If a space *X* has a countable basis for its topology, then *X* is said to satisfy the ***second countability axiom***, or to be ***second-countable***.

Example 1.

1. $\mathbb{R}$ *has a countable basis. It is the collection of all open intervals* (a, b) *with rational end points.*

2. $\mathbb{R}^n$ *has a countable basis. It is the collection of all products of intervals having rational end points.*

3. $\mathbb{R}^{\omega}$ has a countable basis. It is the collection of all product $\prod_{n\in\mathbb{Z}_+} U_n$ where U_n is an open interval with rational end points for finitely many values of n and $U_n = R$ for all values of n.

Theorem 5.1.2.

(i) *A subspace of a first countable space is first countable and a countable product of first countable spaces is first countable.*

(ii) *A subspace of a second countable space is second countable and a countable product of second countable space is second countable.*

Proof.

(i) Let A be a subspace of a first countable space X.

Let $x \in X$.

Let $\mathcal{B}$ be a countable basis for X.

Let $\mathcal{C} = \{B \cap A / B \in \mathcal{B}\}$.

Then $\mathcal{C}$ is a countable basis for the subspace A of X. Therefore, A is first countable.

Let (X_i) be a sequence of first countable spaces.

To prove $\prod X_i$ is first countable.

Let B_i be a countable basis for the space X_i.

Then the collection of all products $\prod U_i$, where $U_i \in B_i$ for finitely many values of *i* is a countable basis for $\prod X_i$. Therefore, $\prod X_i$ is first countable.

(ii) Consider the second countability axiom. Let X be a second countable space.

Let A be a subspace of X.

Let $\mathcal{B}$ be a countable basis for X.

Let $\mathcal{C} = \{B \cap A / B \in \mathcal{B}\}$.

Then $\mathcal{C}$ is a countable basis for the subspace A of X. Therefore, A is second countable.

Therefore, any subspace of a second countable space is second countable.

Let (X_i) be a sequence of second countable spaces.

To prove $\prod X_i$ is second countable.

Let $\mathcal{B}_i$ be a countable basis for the space X_i.

Then the collection of all products $\prod U_i$ where $U_i \in \mathcal{B}_i$ for finitely many values of i is a countable basis for $\prod X_i$. Therefore, $\prod X_i$ is second countable.

Definition. A subset A of a space X is said to be ***dense*** in X if $\bar{A} = X$.

Theorem 5.1.3.

Suppose that *X* has a countable basis. Then:

(a) Every open covering of *X* contains a countable subcollection covering *X*.

(b) There exists a countable subset of *X* that is dense in *X*.

Proof.

Given X as a countable basis.

Let $\{B_n\}$ be a countable basis for the topology on X.

(a) Let $\mathcal{A}$ be an open covering for X.

For each positive integer n for which it is possible to choose an element A_n of A containing the basis element B_n.

That is $B_n \subset A_n$

Let $\mathcal{A}' = \{A_n\}$, then clearly $\mathcal{A}'$ is the countable collection of open subsets of X.

To prove $X = \cup A_n$. Trivially, $\cup A_n \subset X$ ————-(1)

Let $x \in X$

$\Rightarrow x \in A$ for some $A \in \mathcal{A}$.

There exists $B_n \in \{B_n\}$ such that $x \in B_n \subset A$.

Since $B_n \subset A_n$

$\Rightarrow x \in \cup A_n$.

Therefore, $X \subset \cup A_n$ ————-(2).

From (1) and (2) we get, $X = \cup A_n$.

Therefore, $\mathcal{A}'$ is a countable subcollection covering X.

(b) For each nonempty basis element Bn, choose a point $x_n \in B_n$.

Let D be the set consisting of the point x_n.

Clearly, D is the countable subset of X.

Claim : $\overline{D} = X$

Clearly, $\overline{D} \subset$ X.

To prove $X \subset \overline{D}$.

Let $x \in X$.

Let U be a neihbourhood of x.

Then there exists B_n such that $x \in B_n \subset U$.

Now, $x_n \in B_n, x_n \in D$

$\Rightarrow x_n \in B_n \cap D$

$\Rightarrow B_n \cap D \neq \emptyset$

$\Rightarrow x \in D$.

Therefore, $x \subset \overline{D}$. Hence $\overline{D} = X$.

Therefore, D is dense in X.

Definition.

A space for which every open covering contains a countable subcovering is called a ***Lindelof space***. A space having a countable dense subset often said to be ***separable***.

Example 3.

The space $\mathbb{R}_l$ satisfies all the countability axioms but the seconds or $\mathbb{R}_l$ topology is first countable but not second countable.

Proof.

Let $x \in R_l$, the set of all elements of the form $[x, x + \frac{1}{n})$ is a countable basis at x and it is easy to see that the rational number of dense in$\mathbb{R}_l$. Hence it is first countable.

To show $\mathbb{R}_l$ is not second countable.

Let $\mathcal{B}$ be a basis for $\mathbb{R}_l$.

Choose for each x, an element B_i of $\mathcal{B}$ such that $x \in B_x \subset [x, x+1)$.

If $x \neq y$, then $B_x \neq B_y$.

Since $x = inf\ B_x$ and $y = inf\ B_y$.

Therefore, $\mathcal{B}$ must be countable.

Therefore, it does not satisfy the second countability axiom.

Example 4.

The product of two Lindelof spaces need not be Lindelof.

(or)

$\mathbb{R}_l$ is Lindelof but the product $\mathbb{R}_l \times \mathbb{R}_l$ is not Lindelof.

Proof.

The space $\mathbb{R}_l^2$ has basis of all sets of the form $[a, b) \times [c, d)$.

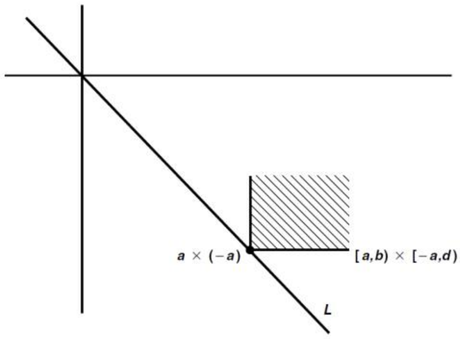
We show that it is not Lindelof.

Consider a subspace $L = \{x \times (-x) / x \in \mathbb{R}_l\}$ and L is closed in $\mathbb{R}_l^2$

Let us cover $\mathbb{R}_l^2$ by the open set $\mathbb{R}_l^2 - L$ and by all elements of the form $[a, b) \times [-a, d)$.

Each of these open sets intersects L in atmost one point.

Since L is uncountable, no countable subcollection covers $\mathbb{R}_l^2$.



Therefore, $\mathbb{R}_l^2$ is not Lindelof.

The subspace of a Lindelof space need not be Lindelof.

The ordered square, I_0^2 is compact.

Therefore, it has a countable subcover.

Therefore, it is Lindelof trivially.

Now, consider the subspace $A = I \times (0,1)$ of I_0^2.

It is not Lindelof.

For, A is the union of disjoint sets, $U_x = \{x\} \times (0,1), x \in I$ each of which is open in A.

This collection of sets is uncountable and no proper subcollection covers A.

It is not Lindelof.

Note: $\mathbb{R}_l^2$ is called sorgenfrey plane.

Example 5. *A subspace of a Lindelof space need not be Lindelof.*

Proof.

The ordered square I_0^2 is compact; therefore, it is Lindelof, trivially.

However, the subspace $A = I \times (0,1)$ is not Lindelof.

For A is the union of the disjoint sets $Ux = \{x\} \times (0,1)$, each of which is open in A.

This collection of sets is uncountable, and no proper subcollection covers A.

5.2. The Separation Axioms

Recall that a space X is said to be ***Hausdorff*** if for each pair x, y of distinct points

of X, there exist disjoint open sets containing x and y, respectively.

Definition.

Suppose that one-point sets are closed in X. Then X is said to be ***regular*** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

The space X is said to be ***normal*** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

Note. It is clear that a regular space is Hausdorff, and that a normal space is regular.

The three separation axioms are illustrated in Figure 5.2.1.

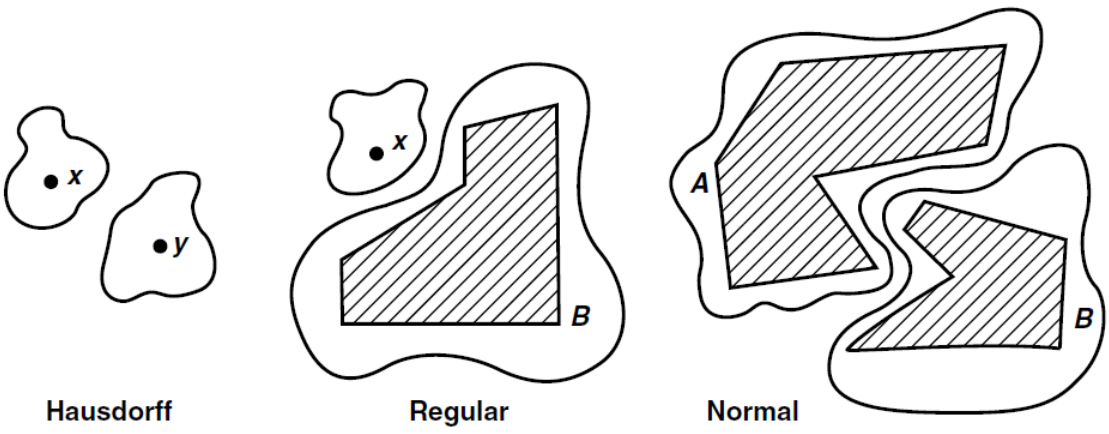


Figure 5.2.1

Lemma 5.2.1.

Let X be a topological space. Let one-point sets in X be closed.

(a) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighbourhood V of x such that $\bar{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.

Proof.

(a) First assume X is regular.

Given a point x and a neighbourhood U of x.

To prove there exists a neighbourhood V of x such that $V \subset U$.

Let $B = X - U$.

Then B is closed in X.

Also $x \notin B$.

Therefore, by hypothesis, there exists disjoint open sets V and W containing x and B respectively.

Therefore, the set V is disjoint from B.

Since if $y \in B$ the set W is a neigbourhood of x such that $V \subset U$.

To prove X is regular.

Suppose the closed set B not containing x be given. Then $x \in U$.

By hypothesis, there is a neighbourhood V of x such that $V \subset U$.

Therefore, the open sets V and $X - V$ are disjoint open set containing x and B respectively.

Hence X is regular.

(b) Suppose that X is normal.

Given a closed set A and an open set U containing A.

Let $B = X - U$.

Since U is open, B is closed in X.

Also, we have A is closed in X.

Since X is normal, there exist disjoint open sets V and W containing A and b respectively.

V is disjoint from W.

Therefore, $\bar{V}$ is disjoint from V.

Therefore, $\bar{V} \subset U$.

Conversely, suppose given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset A$.

To prove that X is normal.

Let $U = X - B$ is an open set containing A.

By hypothesis, there exists an open set V containing A such that $\bar{V} \subset$ U.

Therefore, the open set V and X $- \bar{V}$ are disjoint open set containing A and B respectively.

Also, given that the one-point sets are closed in X.

Therefore, X is normal.

Theorem 5.2.2.

(a) A subspace of a Hausdroff space is Hausdroff. A product of Hausdroff space is Hausdroff.

(b) A subspace of a regular space is regular. A product of a regular space is regular.

Proof.

(a) First let us prove the product of two *H*ausdroff space is *H*ausdroff.

Let X_1 and X_2 be two *H*ausdroff spaces.

To prove $X_1 \times X_2$ is *H*ausdroff space.

That is to prove for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of $X_1 \times X_2, x \neq y$, there exists a neighbourhood U and V of (x_1, x_2) and (y_1, y_2) such that $U \cap V = \emptyset$.

Here $x_1 \in X_1, x_2 \in X_2, y_1 \in X_1, y_2 \in X_2$.

$x \neq y \Rightarrow (x_1, x_2) \neq (y_1, y_2)$

$\Rightarrow x_1 \neq y_1 \; or \; x_2 \neq y_2$.

We take $x_1 \neq y_1$.

Since X_1 is a Hausdroff space, two point $x_1 \neq y_1$ of X_1, there exists a neighbourhood U_1 and U_2 of x_1 and y_1 such that $U_1 \cap U_2 = \emptyset$.

Consider $U_1 \times X_2$ and $U_2 \times X_2$.

Since U_1, U_2, X_2 are open, $U_1 \times X_2$ and $U_2 \times X_2$ are open.

Also, $(x_1, x_2) \in U_1 \times X_2$ and $(y_1, y_2) \in U_2 \times X_2$.

Since $U_1 \cap U_2 = \emptyset, (U_1 \times X_2) \cap (U_2 \times X_2) = \emptyset$.

Thus $U_1 \times X_2$ is a neighbourhood of x_1, x_2 and $U_2 \times X_2$ is a neighbourhood of y_1, y_2 with $(U_1 \times X_2) \cap (U_2 \times X_2) = \emptyset$.

Next to prove subspace of a Hausdroff space is Hausdroff.

Let X be a Hausdroff space.

Let Y be a subspace of X.

To prove Y is Hausdroff.

Let $y_1 \neq y_2$ be two points of Y. Then $y_1, y_2 \in X$.

Since X is Hausdroff, there exists a neighbourhood U_{y_1} and U_{y_2} of y_1 and y_2 in X such that $U_{y_1} \cap U_{y_2} = \emptyset$.

Let $V_{y_1} = U_{y_1} \cap$ Y and $V_{y_2} = U_{y_2} \cap$ Y .

Clearly, V_{y_1} and V_{y_2} are neighbourhood of y_1 and y_2 in Y.

Also, $V_{y_1} \cap V_{y_2} = (U_{y_1} \cap Y) \cap (U_{y_2} \cap Y)$

$= (U_{y_1} \cap U_{y_2}) \cap$ Y.

$= \emptyset \cap Y$

$= \emptyset$.

Therefore, Y is Hausdroff.

(b) Let X be a regular space.

Let Y be a subspace of a regular space X.

Then one point sets are closed in Y.

Let x be a point of Y.

Let B be a closed set in Y not containing the point x.

Now, $\bar{B} \cap Y = B$ where $\bar{B}$ denotes the closure of B in X.

Therefore, $x \notin \bar{B}$.

So, using regularity of X we can choose disjoint open sets U and V of X containing x and $\bar{B}$ respectively.

Then $U \cap Y$ and $V \cap Y$ are disjiont open sets containing X and B respectively.

Therefore, Y is regular.

That is the subspace of X is regular.

That is the subspace of X is regular.

Now, to prove product of a regular space is regular.

let $\{X_\alpha\}$ be a family of regular spaces.

Let $X = \prod X_\alpha$.

By (a) part, X is Hausdroff. So that one-point sets are closed in X.

Let $x = (X_\alpha) \in X$.

Let U be a neighbourhood of x in X.

Choose a basis element $\prod U_x$ about x contained in U.

Then U_α is a neighbourhood of x_α in X_α and each X_α is regular.

Choose for each α, the neighbourhood V_α of x_α such that $V_\alpha \subset U_\alpha$. If it happens that $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$.

Then $V = \prod V_\alpha$ is a neighbourhood of x in X.

Since $\bar{V} \prod \bar{V_\alpha}$.

By a theorem, it follows that, $\bar{V} \subset \prod U_\alpha \subset U$.

That is$\bar{V} \subset U$.

Hence by lemma, X is regular.

That is $\prod X_\alpha$ is regular.

5.3. Normal Space

Theorem 5.3.1.

Every regular space with a countable basis in normal.

Proof.

Let X be a regular space with a countable basis $\mathfrak{B}$

Prove that X is normal.

Let $A\ and\ B$ be disjoint closed subsets of X.

Now $A \cap \bar{B} = A \cap B = \emptyset$

∴ Any point of A is not a limit point of B.

Hence each point x of A has a neighbourhood U not intersecting B.

Since X is regular, we can choose a neighbourhood $V\ of\ x$, whose closure lies in U.

Now choose a basis element of $\mathfrak{B}$ containing x and contained in V.

By choosing such a basis element for each $x \in A$, we construct a countable covering of A by open sets whose closures do not intersect B.

Since this covering of A is countable, we can index it with positive integets.

Let us denote it by $\{U_n\}$

Similarly, we can choose a countable collection $\{V_n\}$ of open sets covering B. Such that each set $\overline{V_n}$ is disjoint from A.

The sets $U = \bigcup_{n\in\mathbb{Z}_+} V_n$ are open sets containing A and B respectively. But they need not be disjoint.

Given n, define $U_n' = U_n - \bigcup_{i=1}^{n} \overline{V_\iota}$ and

$$\overline{V}_n' = V_n - \bigcup_{i=1}^{n} \overline{U}_\iota$$

Since each set U_n' is the difference of open sets U_n and a closed set $\bigcup_{i=1}^{n} \overline{V_\iota}$, U_n' is open, similarly each set V_n' is open,

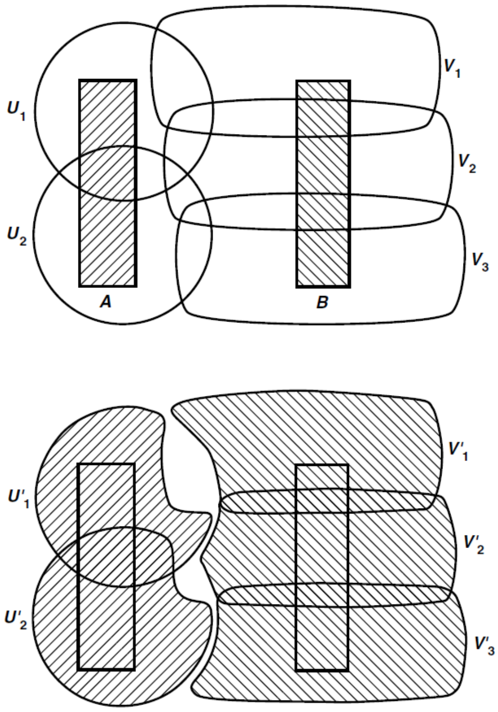


Figure 5.3.1

Claim $\{U_n'\}$ covers A.

Let $x \in A$, then $\alpha \in U_n$, for some n

Similarly each set $\overline{V_\iota}$ is disjoint from A.

$\therefore x \notin \overline{V_\iota} \quad \forall i$

$\therefore x \notin \bigcup_{i=1}^{n} \overline{V_\iota}$

$\therefore x \in U_n - \bigcup_{i=1}^{n} \overline{V_\iota}$

$\Rightarrow x \in U_n'$

$\therefore \{U_n'\}$ covers A.

Similarly $\{V_n'\}$ covers B.

Let $U' = \bigcup_{n\in\mathbb{Z}_+} U'_n$ and $V' = \bigcup_{n\in\mathbb{Z}_+} V'_n$

Then U' and V' are open sets containing A and B respectively.

Claim U' and $V' = \emptyset$.

Assume that $U' \cap V' = \emptyset$.

Let $x \in U' \cap V'$

$\Rightarrow x \in U'$ and $x \in V'$

$\Rightarrow x \in U'_j$ and $x \in V'_k$ for some j and k.

Suppose $j \leq k$.

Now, $x \in U'_j \quad \Rightarrow x \in U_j$ ------------(1)

Now, $x \in V'_k \quad \Rightarrow x \notin \bigcup_{i=1}^{k} \overline{V_\iota}$

$\Rightarrow x \notin \overline{U_\iota} \quad \forall i = 1,2, \dots \dots, k.$

In particular, $x \in \overline{U_J}$ $\quad [\therefore j \leq k]$

$\Rightarrow x \notin U_j$ -----------(2)

∴ equation (2) contratics equation (1)

Also similar contradiction arises if $j \geq k$.

∴ Our assumption is wrong.

Hence $U' \cap V' = \emptyset$.

∴ U' *and* V' are disjoint open sets containing

A and B respectively.

Hence X is normal.

Theorem 5.3.2.

Every metrizable space is normal.

Proof.

Let X be a metrizable space with metric d.

Let A and B be disjoint closed sets in x, for each $a \in A$, we can choose ε_a so that $B(a, \varepsilon_a)$ does not intersect B.

Similarly, for each $b \in B$, we can choose ε_b so that $B(b, \varepsilon_b)$ does not intersect A.

Define $U = \bigcup_{a \in A} B\left(a, \frac{\varepsilon_a}{2}\right)$ and $V = \bigcup_{b \in B} B\left(b, \frac{\varepsilon_b}{2}\right)$

Then U and V are open sets containing A and B respectively.

Claim $U \cap V = \emptyset$.

Assume that $U \cap V \neq \emptyset$

Let $Z \in U \cap V$

$\Rightarrow Z \in U$ and $Z \in V$.

$\Rightarrow Z \in B\left(a, \frac{\varepsilon_a}{2}\right)$, for some $a \in A$ and $Z \in B\left(b, \frac{\varepsilon_b}{2}\right)$ for some $b \in B$.

$\Rightarrow d(a, z) < \frac{\varepsilon_a}{2}$ and $d(b, z) < \frac{\varepsilon_b}{2}$

$\therefore d(a, b) \leq d(a, z) + d(z, b)$

$d(a, b) < \frac{\varepsilon_a}{2} + \frac{\varepsilon_b}{2}$.

If $\varepsilon_a < \varepsilon_b$, then $d(a, b) < \frac{\varepsilon_b}{2} + \frac{\varepsilon_b}{2}$

$$d(a, b) < \varepsilon_b$$

$\Rightarrow a \in B(b, \varepsilon_b)$

Similarly, If $\varepsilon_b < \varepsilon_a$, then $d(a,b) < \frac{\varepsilon_a}{2} + \frac{\varepsilon_a}{2}$

$$d(a,b) < \varepsilon_a$$

$$\Rightarrow b \in B(a, \varepsilon_a)$$

∴ We get a contradiction in both cases.

∴ Our assumption is wrong.

Hence $U \cap V = \emptyset$

∴ U and V are disjoint open set containing A and B respectively.

Hence X is normal.

Theorem 5.3.3.

Every compact Hausdorff space is normal.

Proof.

Let X be a compact Hausdarff space.

To prove X is regular.

Let $x \in X$ and let B be a closed set disjoint from x.

Here B is a closed subset of the compact space X.

Then B is compact.

We know that "*Lemma:* If Y is a compact subspace of the Hausdarff space X and x_0 is not in Y. Then there exist disjoint open set U and V of X containing x_0 and Y respectively"**.**

By the above lemma there exists disjoint open set U and V containing x and B respectively.

$\therefore X$ is regular.

Now to prove X is normal.

Let A and B be disjoint closed sets in X.

Let $a \in A$

Then B is a closed set disjoint from a.

Since X is regular, $\exists$ disjoint open sets U and V of X containing a and B respectively.

Hence for each $a \in A$ we can choose disjoint open sets U_a and V_b containing a and B respectively. Consider the collection, $\{U_a / a \in A\}$ this collection is a covering of A by sets open in X.

Since A is closed subset of the compact space X, A is compact.

$\therefore A$ can be covered by finitely many sets $U_{a_1}, U_{a_2}, \dots \dots, U_{a_n}$.

i.e., $\bigcup_{i=1}^{n} U_{a_i} \supset A$

let $U = U_{a_1} \cup U_{a_2} \cup \dots \dots \dots \cup U_{a_n}$ and $V = V_{a_1} \cap V_{a_2} \cap \dots \dots \cap V_{a_n}$.

Then U and V are open sets containing A and B respectively.

Claim $U \cap V = \emptyset$

Let $Z \in U$, then $Z \in U_{a_i}$ for some a_i

$\Rightarrow Z \notin V_{a_i}$

$\Rightarrow Z \in V$

$\therefore U \cap V = \emptyset$

$\therefore U$ and V are disjoint open sets containing A and B respectively.

Hence X is normal.

Theorem 5.3.4.

*Every well-ordered set X is Normal in the order topology***.**

Proof.

Let X be a well-ordered set.

Step 1:

First, we prove the following resul: "Every interval of the form $(x, y]$ is open in X".

If X has a largest element and y is that element, then $(x, y]$ is a basis element about y.

If y is not the largest element of X. Then, $(x, y] = (x, y')$ where y' is the immediate successor.

$\therefore$ $(x, y]$ is open in X.

Step 2:

Now, we prove that X is normal.

Let A and B be disjoint closed subsets of X.

Case (i). Suppose that A and B do not contain the smallest element a_0 of X.

Then $A \cap \bar{B} = A \cap B = \emptyset$

$\therefore$ for each $a \in A$, there exists a basis element about a disjoint from B.

This basis element contains some interval of the form $(x, a]$. [since a is not a smallest element]

Hence, we can choose for each $a \in A$, such interval $(x_a, a]$ disjoint from B.

Similarly, we can choose for each $b \in B$ an interval $(y_b, b]$ disjoint from A.

Let $U = \bigcup_{a \in A}(x_a, a]$ and $V = \bigcup_{b \in B}(y_b, b]$

By step 1, the interval of the form $(x, y]$ is open in X.

$\therefore$ U and V are open sets containing A and B respectively.

Claim $U \cap V = \emptyset$.

Assume that $U \cap V \neq \emptyset$.

Let $z \in U \cap V$, then $z \in U$ and $z \in V$.

$\Rightarrow z \in (x_a, a]$ for some $a \in A$ and $z \in (y_b, b]$ for some $b \in B$

$\Rightarrow z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and $b \in B$.

Let $a < b$,

If $a \leq y_b$, then $(x_a, a]$ and $(y_b, b]$ are disjoint.

If $a > y_b$ then $a \in (y_b, b]$, where $a \in A$.

$\Rightarrow A \cap (y_b, b] \neq \emptyset$

$\therefore$ We get a contradiction in both cases.

$\therefore$ Our assumption is wrong.

$\therefore U \cap V = \emptyset$.

Hence U and V are disjoint open sets containing A and B respectively.

$\therefore$ X is normal.

Case (ii). Suppose that A contains the smallest element a_0 of X.

Then the set $\{a_0\}$ is both open and closed in X.

Since $\{a_0\}$ is open, $A - \{a_0\}$ is closed in X.

Also, $A - \{a_0\}$ and B are disjoint closed subsets of X.

By case (i), ∃ disjoint open sets U and V containing $A - \{a_0\}$ and B respectively.

Then $U \cap \{a_0\}$ and V disjoint open sets containing A and B respectively.

Hence by both cases X is normal.

Hence the theorem.

Example 2.

The product space $S_\Omega \times \overline{S_\Omega}$ is not normal.

Solution.

Consider the well-order set $\overline{S_\Omega}$ in the order topology and consider the subset S_Ω in the subspace topologies which is same the order topology.

We know that, every well-ordered set is normal in the order topology.

∴ S_Ω and $\overline{S_\Omega}$ are normal.

We prove that the product space $S_\Omega \times \overline{S_\Omega}$ is not normal.

This example serves three purposes.

(i) *A regular space need not be normal.*

For,

S_Ω and $\overline{S_\Omega}$ are normal.

$\Rightarrow S_\Omega$ and $\overline{S_\Omega}$ are regular.

$\Rightarrow S_\Omega \times \overline{S_\Omega}$ is regular, but not normal.

(ii) *A subspace of the normal space is not normal*

For,

$\overline{S_\Omega} \times \overline{S_\Omega}$ is a compact Hausdarff space.

$\Rightarrow \overline{S_\Omega} \times \overline{S_\Omega}$ is normal

But the subspace $\overline{S_\Omega} \times \overline{S_\Omega}$ is not normal.

(iii) *The product of two normal spaces need not be normal.*

Consider, the space $\overline{S_\Omega} \times \overline{S_\Omega}$ and its diagonal

$\Delta = \{x \times x / x \in \overline{S_\Omega}\}$

Claim Δ is closed in $\overline{S_\Omega} \times \overline{S_\Omega}$

$i.e.$ To Prove $(\overline{S_\Omega} \times \overline{S_\Omega})\setminus\Delta$ is open in $\overline{S_\Omega} \times \overline{S_\Omega}$

Let $(x, y) \in (\overline{S_\Omega} \times \overline{S_\Omega})\setminus\Delta$

Then $x \neq y$ in $\overline{S_\Omega}$

Since $\overline{S_\Omega}$ is Hausdarff, $\exists$ disjoint nbd U and V containing x and y respectively.

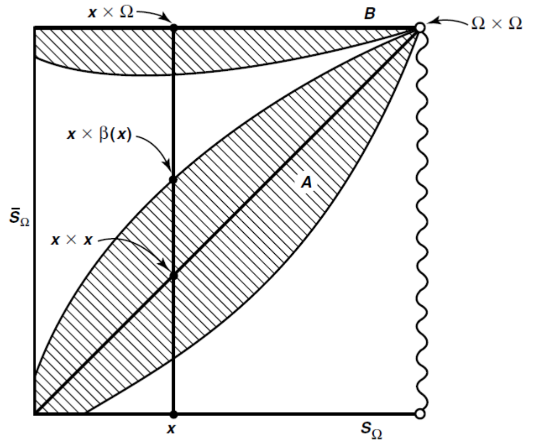


Figure 5.3.2

Since $U \cap V = \emptyset$,

$(x, y) \in U \times V \subset (\overline{S_\Omega} \times \overline{S_\Omega})\setminus\Delta$

$\Rightarrow (\overline{S_\Omega} \times \overline{S_\Omega})\setminus\Delta$ *is open*

$\Rightarrow \Delta$ is closed in $\overline{S_\Omega} \times \overline{S_\Omega}$.

Then in the subspace $S_\Omega \times \overline{S_\Omega}$

$A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$ is closed in $S_\Omega \times \overline{S_\Omega}$

Hence $A = \Delta(\Omega \times \Omega)$

Now, Let $B = S_\Omega \times \{\Omega\}$

Since B is a slice in the product space, B is closed in $S_\Omega \times \overline{S_\Omega}$

$\therefore$ A and B are disjoint closed subsets of $S_\Omega \times \overline{S_\Omega}$

Assume that there exists disjoint open sets U and V in $S_\Omega \times \overline{S_\Omega}$ containing A and B respectively.

Let $x \in S_\Omega$.

Consider the vertical slice $x \times \overline{S_\Omega}$

We prove that there is some point B, with $x < \beta < \Omega$ such that $x \times \beta$ lies outside U.

Suppose that U contains all the points $x \times \beta$ for $x < \beta < \Omega$

Then top point $x \times \Omega$ of the slice $x \times \overline{S_\Omega}$ is the limit point of U.

But $x \times \Omega \in B \subset V$.

i.e., V is a nbd of $x \times \Omega$ which does not intersection U $[\because U \cap V = \emptyset]$

$\Rightarrow x \times \Omega$ is not a limit point of U.

This is a contradiction.

Hence there is some point β with $x < \beta < \Omega$ such that $x \times \beta$ lies outside U.

Let $\beta(x)$ be the smallest element of S_Ω as follows

Let x_1, be any point of S_Ω

Let $x_2 = \beta(x_1), \quad x_3 = \beta(x_2) \dots \dots$ and In general $x_{n+1} = \beta(x_n)$

Since $\beta(x) > x, \forall x$. We have $x_1 < x_2 < x_3 < \cdots \ldots \ldots$

$\therefore \{x_n\}$ is monotonically increasing sequence in S_Ω and the set $\{x_n\}$ is a countable subset of S_Ω.

We know that, A countable subset of S_Ω has an upper bound in S_Ω.

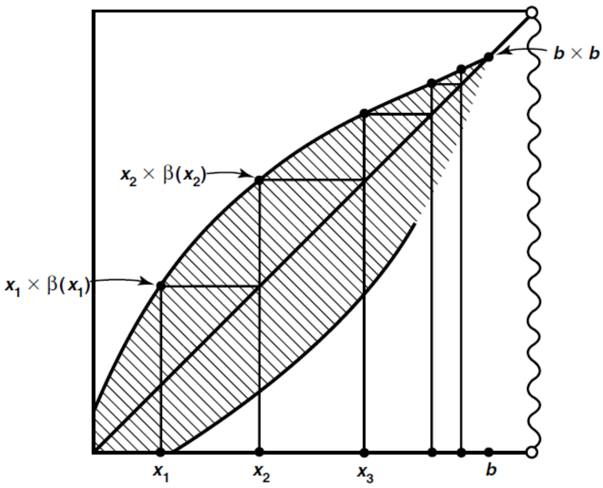
$\therefore$ The set $\{x_n\}$ has an upper bound in S_Ω.

Let $b \in S_\Omega$ be the least upper bound of the set $\{x_n\}$.

Since the sequence (x_n) monotonically increasing $(x_n) \to b$

But $x_{n+1} = \beta(x_n)\ \forall n,\ \left(\beta(x_n)\right) \to b$

Then $\left(x_n \times \beta(x_n)\right) \to b \times b$ in the product space. -------(1)



Now, $b \times b \in A \subset U$.

i.e., U is a nbd of $b \times b$.

But by construction $x_n \times \beta(x_n) \notin U \quad \forall n$

$\Rightarrow \left(x_n \times \beta(x_n)\right) \nrightarrow b \times b$

--------(2)

$\therefore$(2) contradics equation (1),

$\therefore$ Our assumption is wrong.

Hence there is no disjoint open sets U and V in $S_\Omega \times \overline{S_\Omega}$ containing A and B respectively.

$\Rightarrow S_\Omega \times \overline{S_\Omega}$ is not normal.

5.4. The Urysohn Lemma

Theorem 5.4.1.(The Urysohn Lemma)

Let X be a normal space. Let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line then there exists a continuous map $f: X \to [a, b]$ such that $f(x) = a.$ $\forall x \in A$ and $f(x) = b$ $\forall x \in B$.

Proof.

Since $[a, b]$ is homeomorphic to the interval $[0,1]$, it is sufficient to consider the case where the interval in the Question is the interval $[0,1]$.

Step 1: Let P be the set of all rational numbers in the interval $[0,1]$.

We define, for each $p \in P$ an open set U_p of in such a way that when ever $p < q$, we have $\overline{U}_P \subset U_q$.

Thus, the sets U_p will be simply ordered by inclusion in the same way their subscribes are ordered by the usual ordering in the real line.

Since P is countable, we can use induction to define the sets U_{ps}.

Arrange the elements of P in an infinite sequence in some way.

For convenience, let us suppose that the numbers 1 and 0 are the first two elements of the sequence.

Now, we define the sets U_p as follows.

First define $U_1 \ = X - B$

Since $A \cap B = \emptyset$, U_1 is an open set containing the closed set A.

Since X is normal, we can choose an open set U_0

Such that $A \subset U_0$ and $\overline{U_0} \subset U_1$

In general, Let P_n denote the set consisting of the first n rational numbers in the sequence.

Suppose that U_P is defined far all rational numbers P belonging to the set P_n satisfying the condition $p < q \Rightarrow \overline{U}_P \subset U_q$ ---------(*)

Let r denote the next rational number in the sequence.

Now, we define U_r,

Let $P_{n+1} = P_n \cup \{r\}$

P_{n+1} is a finite subset of the interval $[0,1]$ and it has a simple ordering derived from the usual order relation '<' on the real line.

We know that, In a finite simple ordered set every element (other than the smallest and largest) has an immediate predecessor and immediate successor.

The number zero is the smallest element and 1 is the largest element of the simple ordered set $\mathcal{P}_{n+1}$ and r is neither o nor 1.

So r has an immediate predecessor p in P_{n+1} and immediate successor q in P_{n+1}.

The set U_p and U_q are defined already and $\overline{U}_P \subset U_q$ by the induction hypothesis.

Since X is normal we can find an open set U_r in X such that $\overline{U}_p \subset U_r$ and $\overline{U}_r \subset U_q$.

Now, we shall prove that equation (*) hold, for every pair of elements of P_{n+1}

If both elements lie in P_n (*) holds for the induction hypothesis.

If one of them is r and the other is a point s in P_n. Then either $s \leq p$ in which case $\bar{U}_s \subset U_p \subset \bar{U}_p \subset U_r$ (or) $s \leq q$ in which case $\bar{U}_r \subset U_q \subset \bar{U}_q \subset U_s$

Thus, ever pair of elements of P_{n+1}. The relation (*) holds.

By induction, we have defined U_p for all $p \in P$.

Step 2:

In step 1, we defined U_P for all rational numbers p in the interval $[0,1]$.

Now, we extend this definition to all rational numbers p in R by defining $U_P = \emptyset$ if $p < 0$ and $U_p = X$ if $p > 1$.

Then for every pair of rational numbers $p \neq q, p < q \Longrightarrow \bar{U}_p \subset U_q$

Step 3:

Let $x \in X$.

Let $Q(x) = \{p: x \in U_P\}$

Since $U_p = \emptyset$ if $p < 0, Q(r)$

Since $U_p = X$ if $p > 1, Q(r)$ contains every $x \in U_p$

$\therefore Q(x)$ is bounded below and its lower bounded is the point of the interval $[0,1]$ Define $f(x) = inf\ \ Q(x) = inf\{p: x \in U_P\}$

Then f is the function from X into $[0,1]$

Step 4:

Now, we shall prove that f is the desired function.

Claim: $f(x) = 0$ if $x \in A$

Let $x \in A$, since $A \subset U_0, x \in U_0$

$\Rightarrow x \in U_P$ if $p \geq 0$

$\Rightarrow Q(x)$ equals the set of all non-negative rational numbers.

$\Rightarrow f(x) = inf\ Q(x) = 0$

$\therefore f(x) = 0$ if $x \in A$

Claim: $f(x) = 1$ if $x \in B$

Let $x \in B$, Then $x \notin X \backslash B$

$\Rightarrow x \notin U$ [since $U_i x \backslash B$]

Since $p < q$, we have $\bar{U}_P \subset U_q$

Then $x \notin U_1 \Rightarrow x \notin U_P$ if $p \leq 1$

$\Rightarrow Q(x)$ consists of all rational numbers less than or equal to one.

$\Longrightarrow f(x) = inf\ Q(x) = 1$

$\therefore f(x) = 1$ if $x \in B$.

It remains to prove that f is continuous.

For this, first we prove the following elementary facts:

(i) $x \in \bar{U}_r \Longrightarrow f(x) \leq r$

(ii) $x \in U_r \Longrightarrow f(x) \geq r$

(i) Let $x \in \bar{U}_r$,

Since $p < q \Longrightarrow \bar{U}_P \subset U_q, x \in U_S$ if $r < s$

$\Longrightarrow Q(x)$ contains all rational numbers grater than *r.s*

$\Rightarrow f(x) = inf\ Q(x) \leq r.$

$\therefore x \in \bar{U}_r$

$\Rightarrow f(x) \leq r$

(ii) Let $x \notin U_r$,

Then $x \notin U_0$ if $s < r$

$\Rightarrow Q(x)$ contains no rational number less than r

$\Rightarrow f(x) \geq r$

$\therefore x \notin U_r \Rightarrow f(x) \geq r$

Now, we shall prove the continuity of f

Given a point $x_0 \in X$, and the open interval (c, d) in R containing the point $f(x_0)$.

We shall find the neighborhood U of x_0 such that $f(U) \subset (c, d)$.

Since $C < f(x_0) < d$. We can choose the rational numbers p and q such that $c < p < f(x_0) < q < d$.

Let $U = U_q \backslash U_p$. Then U is an open set prove that U is the desired nbd of x_0. See Figure 5.4.2.

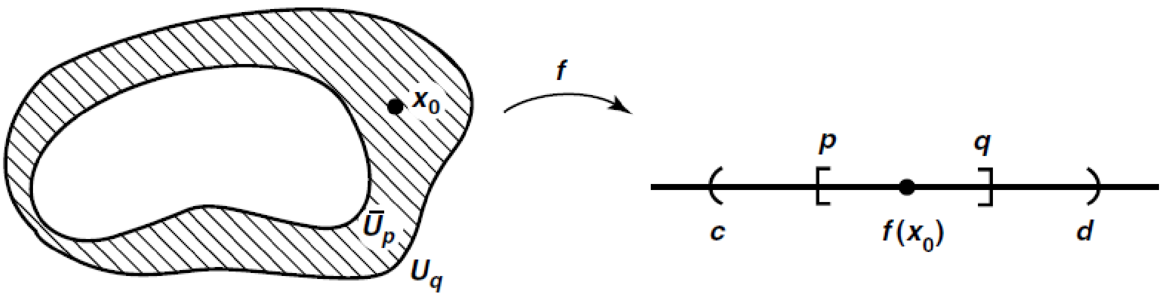


Figure 5.4.2

Claim $x_0 \in U$.

i.e. **To prove** $x_0 \in U_q$ and $x_0 \notin \overline{U}_p$

Assume that $x_0 \notin U_q$

Then $f(x_0) \geq q \quad \big(by\ (ii)\big)$

This is a contradiction

$\therefore x_0 \notin \overline{U}_p$

$\therefore x_0 \in U_q | \overline{U}_p$

$\Longrightarrow x_0 \in U.$

Claim $f(U) \subset (c, d)$

Let $x \in U$. Then $x \in U_q \backslash \overline{U}_q$

$\Longrightarrow x \in U_q$ and $x \notin \overline{U}_p$

$\Longrightarrow x \in \overline{U}_q$ and $x \in U_P$

$\Longrightarrow f(x) \leq q$ and $f(x) \geq p$

$\Longrightarrow p \leq f(x) \leq q$

$\Longrightarrow f(x) \in [p, q]$ which is subset of (c, d)

$\Longrightarrow f(x) \in (c, d)$

Hence $f(U) \subset (c, d)$

$\Longrightarrow f$ is continuous function

Hence the proof.

Definition.

If A and B are two subsets of the topological space X and if there is a continuous function $f: X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B can be ***separated by a continuous function*.**

Remark.

The Urysohn lemma says that if every pair of disjoint closed sets in X can be separated by disjoint open sets, then each such pair can be separated by a continuous function.

The converse is trivial for, if $f: X \to [0,1]$ is the continuous function then $f^{-1}\left(\left[0, \frac{1}{2}\right)\right)$ and $f^{-1}\left(\left(-\frac{1}{2}, 1\right]\right)$ are disjoint open set contains A and B respectively.

Definition (T_5).

A space X is said to be **completely regular** if one-point sets are closed in X and if for each point x_0 of X and each closed set A not containing x_0, there exists a continuous function $f: X \to [0,1]$ such that $f(x_0) = 0$ and $f(AS) = \{1\}$.

Note:

⟹ A normal space is completely regular.

$\Rightarrow$A completely regular space is regular.

Theorem 5.4.2.

(i) *A subspace of a completely regular space is completely regular.*

(ii) *A product of completely regular space is completely regular***.**

Proof.

(i) Let X be a completely regular space and let Y be a subspace of X.

Let x_0 be a point of $Y$and let A be a closed set of Y disjoint from x_0.

Now, $A = \bar{A} \cap Y$,S where $\bar{A}$ denotes the closure of A in X.

$\therefore x_0 \notin \bar{A}$

Since X is completely regular, we can choose a continuous function $f: X \to [0,1]$ such that $f(x_0) = 1$ and $f(\bar{A}) = \{0\}$. Then, the restriction of f to Y is the desire continuous function on Y.

$\therefore$ Y is completely regular.

(ii) Let $X = \prod X_\alpha$ be a product of completely regular space.

To prove X is completely regular.

Let $b = (b_\alpha)$ be a point of X and let A be a closed set of X disjoint from b.

Choose a basis element $\prod U_\alpha$ containing b that does not intersect A.

Then $U_\alpha = X_\alpha$ except for finitely many α say $\alpha = \alpha_1, \alpha_2, \dots \dots, \alpha_n$

Given $i = 1,2, \dots \dots, n$ choose a continuous function $f_i: X_{\alpha_i} \to [0,1]$ such that $f_i(b_{\alpha_i}) = 1$ and $f_i(X \backslash U_{\alpha_i}) = \{0\}$

Let Φ_i maps X continuously into $\mathbb{R}$ and vanish outside $\prod_{\alpha_i}^{-1}(U_{\alpha_i})$

The product $f(x) = \Phi_1(x), \Phi_2(x), \dots \dots., \Phi_n(x)$ is the function such that it equals 1 at b and vanishes outside $\prod U_\alpha$.

$\therefore f$ is the desired continuous function on X.

Hence $X = \prod X_\alpha$ is completely regular.

Note.

(i) The spaces R_l^2 and $S_\Omega \times \overline{S_\Omega}$ are completely regular but not normal.

(ii) A regular space need not be completely regular.

5.5. The Urysohn Metrization Theorem

Theorem 5.5.1(Urysohn Metrization Theorem)

Every regular space X with a countable basis in metrizable.

Proof.

We shall prove that x is metrizable by imbedding X in a metrizable space Y.

i.e. To Prove X is homeomorphic with a subspace of Y.

Let $\{B_n\}$ be a countable basis for X.

Step 1

We prove the following:

"There exists a countable collection of continuous function $f_n: x \to [0,1]$ having the property that given a point $x_0\ of\ X$ and given a nbd U of x_0, there exists an index n such that f_n in positive at x_0 and vanishes outside U".

Let n and m be a pair of indexes for which $\overline{B_n} \subset B_m$. Then $\overline{B_n}$ and $X\backslash B_m$ are disjoint closed subsets of x.

Since X is regular with countable basis, X is normal.

∴ By the Urysohn lemma, We can choose a continuous function $g_{n,m}: X \to [0,1]$ such that $g_{n,m}\overline{(B_n)} = \{1\}\ and\ g_{n,m}(X\backslash B_m) = \{0\}$

Hence far each pair n, m of indices for which $\overline{B_n} \subset B_m$. we can choose a continuous function $g_{n,m}: X \to [0,1]$ such that $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X\backslash B_m) = \{0\}$

Now we shall prove that $\{g_{n,m}\}$ satisfies our requirements.

Let U be a nbd ofx_0 then we can choose a basis element B_m such that $x_0 \in B_m \subset U$.

Since X is regular, we can choose a basis element B_n such that $x_0 \in B_n$ and $\overline{B_n} \subset B_m$

Then far this pair n, m of indices $g_{n,m}$ is defined.

Now, $g_{n,m}(x_0) = 1$ $\quad$ $[\because x_0 \in B_n \subset \overline{B_n}]$

∴ $g_{n,m}$ is positive at x_0.

Let $x \notin U$

Then $x \notin B_m$ $\quad$ $[\because B_m \subset U]$

$\Rightarrow g_{n,m}(x) = 0$

$\Rightarrow g_{n,m}$ vanishes outside U.

Since this collection is indexed by a subset of $Z_+ \times Z_+$, it is countable.

Hence $\{g_{n,m}\}$ satisfies our requirement.

Since this collection is countable, the collection can be re indexed with a positive integer giving us the desired indexed family $\{f_n\}$.

Step 2: (first version of the proof)

Consider $\mathbb{R}^w$ is the product topology, given the function f_n of step q, define a map $F: x \to \mathbb{R}^w$ by $F(x) = (f_1(x), f_2(x) \dots \dots .)$

Claim

F is continuous.

Since $\mathbb{R}^\omega$ has the product topology and each f_n is continuous, F is continuous.

Claim: F is injective.

Let $x \neq y$

Since X is regular, X is Hausdorff

∴ There exists disjoint neighbourhood U_x and U_y of x and y respectively.

Then $x \in U_x$ and $u \notin U_x$,

By step 1, there exists an induced n such that $f_n(x) > 0$ and $f_n(y) = 0$.

$\Rightarrow f_n(x) \neq f_n(y)$

$\Rightarrow F(x) \neq F(y)$

$\Rightarrow F$ is one – one

Hence F is injective.

Now to prove, F is an imbedding of X in $\mathbb{R}^\omega$, we shall prove that F is a homeomorphism of X onto its image, the subspace $Z = F(X)$ of $\mathbb{R}^\omega$.

We know that, F defines a continuous bijection of X with Z.

It remains to prove that $F^{-1}: Z \to X$, is continuous.

i.e, To prove for each open set U in X, $F(U)$ is open in Z.

Let U be open in X.

To prove $F(U)$ is open in Z.

Let z_0 be a point of $F(U)$

We shall find an open set W of Z, such that $z_0 \in W$

Let x_0 be a point of U, such that $F(x_0) = z_0$

Choose an index N for which $f_N(x_0) > 0$ and $f_N(X \setminus U) = 0$.

Consider the open ray $(0, \infty)$ in $\mathbb{R}$

Let $V = \prod_N^{-1}((0, \infty))$

Then V is open in $\mathbb{R}^\omega$.

Let $W = V \cap Z$

Since V is open in $\mathbb{R}^\omega$, $V \cap Z$ is open in Z.

$\therefore$ W is open in Z.

To prove $z_0 \in W \subset F(U)$

Claim 1 $z_0 \in W$

Now $\prod_N(z_0) = \prod_N(F(x_0))$

$= \prod_N(f_1(x_0), f_2(x_0) \ldots \ldots \ldots)$

$= f_N(x_0) > 0$

$\therefore \prod_N(z_0) \in (0, \infty)$

$\Rightarrow z_0 \in \prod_N^{-1}((0, \infty))$

$\Rightarrow z_0 \in V$

Also $z_0 \in Z$

$\therefore z_0 \in V \cap Z$

$\Rightarrow z_0 \in W$

Now claim 2: $W \subset F(U)$

Let z$\in W$

$\Rightarrow$ z $\in V \cap Z$

$\Rightarrow$ z $\in V$ and $z \in Z = F(X)$

$\Rightarrow$ z $\in \prod_N^{-1}\big((0,\infty)\big)$ and $z = F(x)$ for some $x \in X$

$\Rightarrow \prod_N(\text{z}) \in (0,\infty)$ and $Z = F(x)$

Now $\prod_N(\text{z}) = \prod_N\big(F(x)\big)$

$= \prod_N(f_1(x), f_2(x) \dots \dots \dots)$

$\prod_N(\text{z}) = f_N(x)$.

$\therefore \prod_N(\text{z}) = f_N(x)$

$\therefore \prod_N(\text{z}) \in (0,\infty)$

$\Rightarrow f_N(x) \in (0,\infty)$

Since f_N vanishes outside U, we must have x in U

i.e., $x \in U$

$\Rightarrow F(x) \in F(U)$ $\quad\quad [z = F(x)]$

$\Rightarrow Z \in F(U)$

$\therefore W \subset F(U)$

$\therefore F(U)$ is open in Z

$\Rightarrow F^{-1}$ is continuous.

Hence F is a homeomorphism of X enter the impace of $\mathbb{R}^\omega$.

Thus F is a imbedding of X in $\mathbb{R}^\omega$.

Hence X is metrizable.

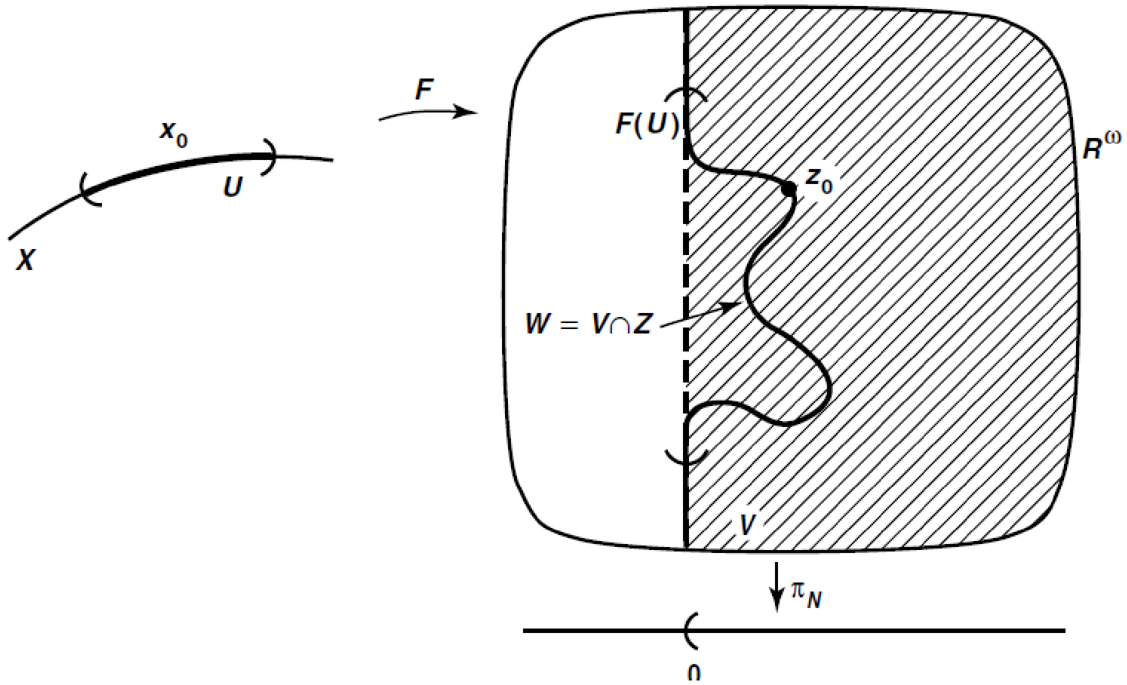


Figure 5.5.1s

Step 3 (second version of the proof)

In this version we imbed X in the metric space $(\mathbb{R}^{\omega}, \bar{\rho})$ actually we imbed X in the subspace $[0,1]^{w}$ on which $\bar{\rho}$ equals the metric $\rho(x,y) = lub\{|x_i - y_i|\}$.

We use the countable collection of function $f_n: X \to [0,1]$ constructed in step 1.

But now we impost the additional condition that $f_n(x) \leq \frac{1}{n} \forall x$, this condition is satisfied by just dividing each function f_n by n.

Since $f_n(x) \leq \frac{1}{n} \forall x \in X$.

f_n maps X into $\left[0, \frac{1}{n}\right]$

Define $F: X \to [0,1]^{w}$ by the equation

$F(x) = (f_1(x), f_2(x), \dots \dots \dots)$

Now we shall prove that , F is an imbedding relative to the metric ρ on $[0,1]^{w}$

By the step 2, F is injective.

Also, if we use the product topology on $[0,1]^{w}$, F carries open set of X, onto open set of $Z = F(X)$

This statement remains true if one passes to the finer (larger) topology on $[0,1]^\omega$ induced by the metric ρ

It remains to prove that F is continuous.

Let $x_0 \in X$ and Let $\varepsilon > 0$

To prove the continuity, we shall find the $nbd\ U\ of\ x$, such that $x \in U \Rightarrow \rho\big(F(x), F(x_0)\big) < \varepsilon$

First choose n, larger enough such that $\frac{1}{N} \leq \frac{\varepsilon}{2}$

Since each f_n is continuous, far each $n = 1,2 \dots \dots N$

We can choose a $nbd\ U_n$ of x_0 such that $|f_n(x) - f_n(x_0)| \leq \frac{\varepsilon}{2}\ \forall x \in U_n$

Let $U = U_1 \cap\ U_2 \cap \dots \dots \dots \cap U_N$

Then U is a nbd of x_0.

To prove U is the desired nbd of x_0

Let $x \in U$

Then $x \in U_n, \forall\ n = 1,2, \dots \dots \dots N$

Case (i): Let $n \leq N$

Since $x \in U_n$, $|f_n(x) - f_n(x_0)| \leq \frac{\varepsilon}{2}\ \ \forall\ x \in U$

Case (ii): Let $n > N$

Then $\frac{1}{n} < \frac{1}{N}$

We know that f_n maps X into $\left[0, \frac{1}{n}\right]$

$\therefore f_n(x) \in \left[0, \frac{1}{n}\right]$ and $f_n(x_0) \in \left[0, \frac{1}{n}\right]$

$\Rightarrow |f_n(x) - f_n(x_0)| \leq \frac{1}{n} < \frac{1}{N} < \frac{\varepsilon}{2}$

If $n > N, |f_n(x) - f_n(x_0)| \leq \frac{\varepsilon}{2}$

Then by both cases $\rho\big(F(x), F(x_0)\big) = lun\ \{|F_n(x), \leq \frac{\varepsilon}{2} < \varepsilon\ \forall x\}$

Hence F is continuous.

Thus F is an imbedding of X in $[0,1]^w$

Since $[0,1]^w$ is metrizable, X is metrizable.

Hence the theorem.

Theorem 5.5.2 (Imbedding theorem)

Let X be a space in which one-point sets are closed. Suppose that the collection $\{f_\alpha\}_{\alpha\in J}$ *is an indexed family of continuous function* $f_\alpha: X \to R$ *satisfying the requirement that for each point* x_0, X *and each nbd* U *of* x_0*, there is an index* α *such that* f_α *is positive at* x_0 *and vanishes outside* U*. Then the function* $F: X \to \mathbb{R}^J$ *defined by* $F(x) = \left(f_\alpha(x)\right)_{\alpha\in J}$ *is an imbedding of X in* $\mathbb{R}^J$*. If* f_α*maps* X *into* $[0,1]$ *for each* α *then* F *imbeds* X *in* $[0,1]^J$.

Proof.

Replace n by α and $\mathbb{R}^w$ by $\mathbb{R}^J$ in step 2 in the previous theorem.

Definition.

A family of continuous function that satisfies the Hypothesis of this theorem is said to ***separate points from closed sets*** in X.

Theorem 5.5.3.

A space X is completely regular iff it is homeomorphic to a subspace of $[0,1]^J$ for some J.

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